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主論文

Cohomologies of Lie algebras of vector fields with coefficients in adjoint representations

Hamiltonian Case

By

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Introduction.

Let M be a smooth manifold, and $A(M)$ the infinite dimensional Lie algebra of all smooth vector fields on M. Let A be $A(M)$ or a certain natural subalgebra of it. We are interested in the cohomology $H^*(A\,;\,V)$ of A with coefficients in some representation V, which is an invariant of the Lie algebra A.

In 1968, I. M. Gel'fand and D. B. Fuks began to study the theory of cohomologies of Lie algebras of vector fields. First, they treated the case where $A = A(M)$ and $V = R$ (trivial coefficients). Since then, many mathematicians studied cohomologies in many cases, for instance [2], [4], [6] etc. They also treated the case of nontrivial coefficients.

but restricted themselves to the representations induced from some finite dimensional ones. Their proofs were essentially based upon some finiteness of representations.

Meanwhile, in 1973, F. Takens [7] proved that any derivations of Ⓐ(M) is inner. It means that the first cohomology of Ⓐ(M) with coefficients in its adjoint representation, a natural infinite dimensional representation, is trivial.

In the present paper, the author treats a symplectic manifold (M, ω) and the subalgebra $Ⓐ_\omega(M)$ consisting of hamiltonian vector fields on M in this direction. Then he obtains the following results.

<u>Main Theorem</u>. Let (M, ω) be a connected symplectic manifold, then the first cohomology of $Ⓐ_\omega(M)$ with coefficients in its adjoint representation, is of dimension 1 or 0, that is,

$$\dim H^1(Ⓐ_\omega(M) ; Ⓐ_\omega(M)) = 1 \text{ or } 0.$$

Moreover, $H^1(Ⓐ_\omega(M) ; Ⓐ_\omega(M)) \cong R$ if and only if the symplectic form ω is exact.

Locally, this theorem has a simple feature (Theorem 5) as follows: Let U be a connected and simply connected open set in R^{2n}, with the natural symplectic structure $\omega = \Sigma\, dx_i\, dy_i$, then

$$H^1(Ⓐ_\omega(U)\ ;\ Ⓐ_\omega(U)) \cong R.$$

The proof of Main Theorem can be carried out by elementary calculations. But to make short some part of the proof, we use Weyl's results on representations of the symplectic algebra. The elementary version of that part is outlined also in Section 4.

In §1, we explain some generalities of the first cohomology and symplectic manifolds. In §2, we prove interesting properties (Propositions 1 and 4) of hamiltonian vector fields, which play an important role to prove Theorem 5, a local theorem. Moreover we prove in §2 that a derivation of $Ⓐ_\omega(M)$ is a local operator (Proposition 3). Section 3 is devoted to the study of derivations of $Ⓐ_\omega(M)$ in local. In §4, we complete the proof of Theorem 5. Here we use some knowledge of formal hamiltonian vector fields. Finally in §5, we give the proof of Main Theorem.

The author expresses his hearty thanks to Professors T. Hirai, T. Morimoto and N. Tatsuuma for their kind advices.

§1. Derivations and $H^1(\mathfrak{A};\mathfrak{A})$

All manifolds, vector fields, functions etc. are assumed to be of C^∞-class.

Let $\mathfrak{A}$ be a subalgebra of the Lie algebra $\mathfrak{A}(M)$ of all vector fields on a manifold M, and consider the adjoint representation of $\mathfrak{A}$:

$$(\mathrm{ad}\, X)(Y) = [X, Y] \qquad (X, Y \in \mathfrak{A}),$$

where $[\ ,\ ]$ is the usual bracket operation of vector fields. The cochain complex $\{C^q(\mathfrak{A};\mathfrak{A}),\ d^q\}$ of the Lie algebra $\mathfrak{A}$ with coefficients in its adjoint representation consists of the followings:

$$C^q(\mathfrak{A};\mathfrak{A}) = \{P : \mathfrak{A}\times\cdots\times\mathfrak{A} \longrightarrow \mathfrak{A},\ \text{skew-symmetric } q\text{-linear map}\},$$

and for $P \in C^q$ and $X_1, \ldots, X_{q+1} \in \mathfrak{A}$,

$$(d^q P)(X_1, \cdots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} [X_i, P(X_1, \cdots, \hat{X}_i, \cdots, X_{q+1})]$$
$$+ \sum_{i<j} (-1)^{i+j} P([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{q+1}).$$

The homologies $\{H^q(\mathfrak{A}; \mathfrak{A}), q \geq 0\}$ of this complex are called the cohomologies of $\mathfrak{A}$ with coefficients in its adjoint representation. The one dimensional cohomology $H^1(\mathfrak{A}, \mathfrak{A})$ is interpreted as follows. Since

$$dP(X, Y) = [X, P(Y)] - [Y, P(X)] - P([X, Y]) \qquad (X, Y \in \mathfrak{A}),$$

for $P \in C^1$, we see that $dP = 0$ means that

$$P([X, Y]) = [P(X), Y] + [X, P(Y)],$$

that is, 1-dimensional cocycles are derivations of $\mathfrak{A}$. Moreover since

$$(dQ)(X) = [X, Q] \qquad (X \in \mathfrak{A}),$$

for $Q \in C^0(\mathfrak{A}; \mathfrak{A}) = \mathfrak{A}$, we see that 1-dimensional coboundaries are inner derivations of $\mathfrak{A}$. Thus the first cohomology space $H^1(\mathfrak{A}; \mathfrak{A})$ is the equivalence classes of the algebra $\mathfrak{D}(\mathfrak{A})$ of derivations of $\mathfrak{A}$ modulo its ideal $\mathfrak{D}^i\mathfrak{A}$ of inner derivations, or

$$H^1\mathfrak{A} : \mathfrak{A} \cong \mathfrak{D}(\mathfrak{A})/\mathfrak{D}^i(\mathfrak{A}) .$$

In the following, we consider a smooth symplectic manifold (M^{2n}, ω), and the subalgebra $\mathfrak{A} = \mathfrak{A}_\omega(M)$ of hamiltonian vector fields on M. A symplectic structure is defined on M^{2n} by a nondegenerate closed 2-form ω, that is, $\omega^n = \omega \wedge \cdots \wedge \omega$ is a volume form of M and $d\omega = 0$. A vector field X is called hamiltonian, if it preserves the symplectic form ω, and by definition,

$$\mathfrak{A}_\omega(M) = \{X \in \mathfrak{A}(M) \ ; \ L_X \omega = 0\},$$

where $L_X \omega$ is the Lie derivative of ω by X. To determine the first cohomology, we must study the structures of $\mathfrak{D}_\omega(M) = \mathfrak{D}(\mathfrak{A}_\omega(M))$ and $\mathfrak{D}^i_\omega(M) = \mathfrak{D}^i(\mathfrak{A}_\omega(M))$.

§2. Some properties of $\mathfrak{A}_\omega(M)$ and $\mathfrak{D}_\omega(M)$.

2.1. In the following, we denote by ∂_v the vector field $\partial/\partial v$.

and by $X_{|U}$ the restriction of X on U.

Proposition 1. Let p be a point of a symplectic manifold M^{2n}, and let X be a hamiltonian vector field on M such that $j^2(X)(p) = 0$, that is, the 2-jet of each of the coefficient functions of X is zero at p.

Then, there exist a finite number of hamiltonian vector fields $Y_1, \cdots, Y_\ell, Z_1, \cdots, Z_\ell$ on M, and a neighbourhood U of p in M, such that

$$X_{|U} = \sum_{i=1}^{\ell} [Y_i, Z_i]_{|U} ,$$

and

$$j^1(Y_i)(p) = j^1(Z_i)(p) = 0 \qquad (1 \leq i \leq \ell).$$

Proof. Let U be a simply connected open neighbourhood of p, and let $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ be a symplectic coordinate system around p in U, that is,

$$\omega_{|U} = \sum_{i=1}^{n} dx_i \, dy_i .$$

Since the vector field X is hamiltonian, $L_X \omega = di_X \omega = 0$, where $i_X \omega$ is the interior product of X and ω. Hence the differential form $i_X \omega$ is closed, and so the restriction $i_X \omega|_U$ is exact by Poincare's lemma. With respect to the above local coordinates, X and $i_X \omega$ are written in U as

$$X = \sum_{i=1}^{n} \{f_i(x, y)\partial_{x_i} + g_i(x, y)\partial_{y_i}\},$$
$$i_X\omega = \sum_{i=1}^{n} \{f_i(x, y)\, dy_i - g_i(x, y)\, dx_i\},$$

where f_i and g_i are functions on U. There exists a function $H = H(x, y)$ on U such that

$$i_X\omega = dH = \sum_{i=1}^{n} (H_{x_i}\, dx_i + H_{y_i}\, dy_i).$$

Therefore we have for $i = 1, \cdots\cdots, n$,

$$f_i(x, y) = H_{y_i}, \qquad g_i(x, y) = -H_{x_i},$$

that is,

$$X = \sum_{i=1}^{n} (H_{y_i}\partial_{x_i} - H_{x_i}\partial_{y_i}) \qquad \text{on} \quad U.$$

This function H is uniquely determined up to constants, so that we may put $H(p) = 0$. A function or vector field is called without constant term if it is zero at the origin of the coordinate system.

In global, any function H on M uniquely determines the hamiltonian vector field X on M by the formula $i_X \omega = dH$, because of non-degeneracy of the symplectic form ω. So X may be written as X_H. Then the following formula holds for two functions H and K on M,

$$[X_H, X_K] = X_{-\{H, K\}} .$$

Here $\{H, K\}$ is a function on M called the Poisson bracket of H and K, which is given in U by local coordinates as

$$\{H, K\} = \sum_{i=1}^{n} (H_{x_i} K_{y_i} - H_{y_i} K_{x_i}) .$$

Thus, the proposition follows from the following result on a connected open set (called domain) in a Euclidean space.

q. e. d.

Proposition 2. Let H be a C^{∞}-function on a simply connected domain U in R^{2n} with $j^3(H)(0) = 0$, then there exist a finite number of C^{∞}-functions $K_1, \cdots, K_{\ell}, G_1, \cdots, G_{\ell}$ on U, such that

$$H = \sum_{i=1}^{\ell} \{K_i, G_i\}.$$

and

$$j^2(K_i)(0) = j^2(G_i)(0) = 0 \qquad (1 \leq i \leq \ell).$$

Proof. Since $j^3(H)(0) = 0$, H can be given as a finite sum of functions of the following form:

$$x_1^{\ell_1} \cdots x_n^{\ell_n} y_1^{m_1} \cdots y_n^{m_n} f(x, y)$$

with $\sum_{i=1}^{n} (\ell_i + m_i) \geq 4$, and f a C^{∞}-function on U. Since $\sum_{i=1}^{n} \ell_i \geq 2$, or $\sum_{i=1}^{n} m_i \geq 2$, we may assume $\sum_{i=1}^{n} \ell_i \geq 2$ without loss of generality.

Case 1. The case where $\ell_{i_0} \geq 2$ for some i_0. Let $i_0 = 1$, and put

$$K = x_1^3, \qquad G = x_1^{\ell_1 - 2} \prod_{i=2}^{n} x_i^{\ell_i} y_i^{m_i} g(x, y),$$

where

$$g = \frac{1}{3} \int_0^{y_1} y_1^{m_1} f(x, y)\, dy_1,$$

then we have

$$\begin{aligned} \{K, G\} &= 3\, x_1^{\ell_1} \prod_{i=2}^{n} x_i^{\ell_i} y_i^{m_i} g_{y_1}(x, y) \\ &= x_1^{\ell_1} \cdots x_n^{\ell_n} y_1^{m_1} \cdots y_n^{m_n} f(x, y). \end{aligned}$$

Moreover $j^2(K)(0) = j^2(G)(0) = 0$, because $j^{m_1}(g)(0) = 0$, and

$$\sum_{i=2}^{n} (\ell_i + m_i) + (\ell_1 - 2) + (m_1 + 1) = \sum_{i=1}^{n} (\ell_i + m_i) - 1 \geq 3.$$

<u>Case 2</u>. The case where all $\ell_i \leq 1$. Assume that $\ell_1 = \ell_2 = 1$, then by means of the following symplectic transformation, this case is reduced to Case 1:

$$\begin{cases} x_1' = \sqrt{2}^{-1} (x_1 + x_2), & y_1' = \sqrt{2}^{-1} (y_1 + y_2), \\ x_2' = \sqrt{2}^{-1} (x_1 - x_2), & y_2' = \sqrt{2}^{-1} (y_1 - y_2), \\ x_i' = x_i, & y_i' = y_i \qquad (i \geq 3). \end{cases}$$

q. e. d.

2.2. Proposition 3. Let D be a derivation of $\mathfrak{A}_\omega(M)$. If $X \in \mathfrak{A}_\omega(M)$ is identically zero on some domain U in M, then $D(X)$ vanishes identically on U.

Proof. Assume that there exists a point p in U such that $D(x)(p) \neq 0$. Let V be a simply connected coordinate neighbourhood of p in U. Since $D(X)$ is hamiltonian, using symplectic coordinates around p in V, we can find a function H on V such that $D(X)|_V = X_H$, as in the proof of Proposition I. Since $D(X)(p) \neq 0$, $H_{x_i}(p) \neq 0$ or $H_{y_i}(p) \neq 0$ for some i. We may assume that $H_{x_i}(p) \neq 0$. Let K be a function whose support is contained in V, and equals to y_1^2 in a smaller neighbourhood V' of p. Then we have

$$\{H, K\} = 2y_i H_{x_i} \qquad \text{in } V',$$

and then

$$\{H, K\}_{y_i} = 2H_{x_i} + 2y_i H_{x_i y_i} .$$

Hence

$$X_{-\{H,\ K\}}(p) = -H_{x_i}(p)\partial_{x_i} \neq 0 .$$

On the other hand, since $[X,\ X_K] = 0$ on M,

$$0 = D([X,\ X_K])(p) = [D(X),\ X_K](p) + [X,\ D(X_K)](p)$$

$$= X_{-\{H,\ K\}}(p).$$

This contradicts our assumption.

q. e. d.

Proposition 4. Let D be a derivation of $Ⓐ_\omega(M)$, and X be a hamiltonian vector field on M. If $j^2(X)(p) = 0$ for some point p of M, then $D(X)(p) = 0$.

Proof. We can find, by Proposition 1, a neighbourhood U of p, and hamiltonian vector fields $Y_1, \cdots, Y_\ell$ and $Z_1, \cdots, Z_\ell \in Ⓐ_\omega(M)$ such that

$$X_{|U} = \sum_{i=1} [Y_i,\ Z_i]_{|U},$$

$$j^1(Y_i)(p) = j^1(Z_i)(p) = 0 \qquad (1 \leq i \leq \ell).$$

Then, using Proposition 3, we get

$$D(X)(p) = D(\sum_i [Y_i, Z_i])(p)$$

$$= \sum_i([D(Y_i), Z_i](p) + [Y_i, D(Z_i)](p)) = 0.$$

q. e. d.

Remark 1. Any derivation D of Ⓐ$_\omega(M)$ can be considered as a derivation of Ⓐ$_\omega(U)$ for any open subset U of M.

In fact, for any point p in U, by the proof of Proposition 1, we have a hamiltonian vector field $\tilde{X}$ on M for any $X \in$ Ⓐ$_\omega(U)$ such that $\tilde{X}$ equals to X on some neighbourhood of p. Define D_U by $D_U(X)(p) = D(\tilde{X})(p)$, then $D_U(X)(p)$ is well defined by Proposition 3, and clearly D_U is a derivation of Ⓐ$_\omega(U)$.

§3. Inner derivations of Ⓐ$_\omega(U)$.

3.1. In this section, we fix a simply connected domain U of M, and a coordinate system $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ in U for which $\omega = \Sigma\, dx_i\, dy_i$ in U. The conditions that a vector field X on U

is hamiltonian, is given as follows:

$$\text{(1)} \qquad \begin{aligned} & X = \sum_{i=1}^{n} (f_i(x, y)\partial_{x_i} + g_i(x, y)\partial_{y_i}) \in \mathfrak{A}_\omega(U) \\ \Longleftrightarrow \quad & \partial_{x_j}(f_i) = -\partial_{y_i}(g_j),\ \partial_{y_j}(f_i) = \partial_{y_i}(f_j),\ \partial_{x_j}(g_i) = \partial_{x_i}(g_j) \\ & \qquad\qquad\qquad\qquad\qquad\qquad (1 \leq i, j \leq n). \end{aligned}$$

<u>Theorem 5</u>. Let D be a derivation of the Lie algebra $\mathfrak{A}_\omega(U)$ of hamiltonian vector fields on U.

(i) There exists a unique vector field Z (not necessarily hamiltonian) on U such that

$$D(X) = [Z, X] \qquad\qquad (X \in \mathfrak{A}_\omega(U)).$$

(ii) Z is uniquely expressed as $Z = Z_1 + Z_2$, where $Z_1 \in \mathfrak{A}_\omega(U)$ and for some constant c,

$$Z_2 = c \sum_{i=1}^{n} (x_i \partial_{x_i} + y_i \partial_{y_i}).$$

<u>Note</u>. If $c \neq 0$, Z_2 is not hamiltonian, because $L_{Z_2}\omega = 2c\,\omega$ in U.

Let us call a vector field constant or linear if it has only constant coefficients or linear coefficients respectively.

We construct the vector field Z as a sum of $Z^{(0)}$, $Z^{(1)}$ and $Z^{(2)}$:

$$Z = Z^{(0)} + Z^{(1)} + Z^{(2)}.$$

Here $Z^{(0)}$ is the constant term of Z, and $Z^{(1)}$ is the linear term of Z (containing Z_2), and finally $Z^{(2)}$ is the remaining term with coefficient functions of degree ≥ 2, a hamiltonian field.

3.2. Determination of $Z^{(2)}$.

According to the situation, x_i is denoted by v_i, and y_i by v_{i+n} for $1 \leq i \leq n$. To determine $Z' = Z^{(1)} + Z^{(2)}$, we will use the following equalities,

$$D(\partial_{v_i}) = [Z, \partial_{v_i}] = [Z', \partial_{v_i}] \qquad (1 \leq i \leq 2n).$$

Define for all i and j, the functions f_{ij} etc. on $\mathcal{U}$ as

$$D(\partial_{x_i}) = \sum_{j=1}^{n} (f_{ij}\partial_{x_j} + g_{ij}\partial_{y_j}),$$
$$D(\partial_{y_i}) = \sum_{j=1}^{n} (f'_{ij}\partial_{x_j} + g'_{ij}\partial_{y_j}).$$

It follows from $\left[\partial_{v_\ell}, \partial_{v_m}\right] = 0$ for ℓ, $m = 1, \cdots, 2n$ that

$$0 = D(\left[\partial_{x_\ell}, \partial_{x_m}\right]) = \left[D(\partial_{x_\ell}), \partial_{x_m}\right] + \left[\partial_{x_\ell}, D(\partial_{x_m})\right]$$

$$= \sum_j \{ (\partial_{x_\ell}(f_{mj}) - \partial_{x_m}(f_{\ell j}))\partial_{x_j} + (\partial_{x_\ell}(g_{mj}) - \partial_{x_m}(g_{\ell j}))\partial_{y_j} \},$$

$$0 = D(\left[\partial_{x_\ell}, \partial_{y_m}\right])$$

$$= \sum_j \{ (\partial_{x_\ell}(f'_{mj}) - \partial_{y_m}(f_{\ell j}))\partial_{x_j} + (\partial_{x_\ell}(g'_{mj}) - \partial_{y_m}(g_{\ell j}))\partial_{y_j} \},$$

and that

$$0 = D(\left[\partial_{y_\ell}, \partial_{y_m}\right])$$

$$= \sum_j \{ (\partial_{y_\ell}(f'_{mj}) - \partial_{y_m}(f'_{\ell j}))\partial_{x_j} + (\partial_{y_\ell}(g'_{mj}) - \partial_{y_m}(g'_{\ell j}))\partial_{y_j} \}.$$

Therefore we have for all j, ℓ, m,

$$(2) \quad \partial_{x_\ell}(f_{mj}) = \partial_{x_m}(f_{\ell j}), \quad \partial_{y_m}(f_{\ell j}) = \partial_{x_\ell}(f'_{mj}), \quad \partial_{y_\ell}(f'_{mj}) = \partial_{y_m}(f'_{\ell j}),$$

$$(3) \quad \partial_{x_\ell}(g_{mj}) = \partial_{x_m}(g_{\ell j}), \quad \partial_{y_m}(g_{\ell j}) = \partial_{x_\ell}(g'_{mj}), \quad \partial_{y_\ell}(g'_{mj}) = \partial_{y_m}(g'_{\ell j}).$$

Since U is simply connected, there are unique functions φ_j and $\psi_j (1 \leq j \leq n)$ up to constants on U such that

$$\partial_{x_i}(\varphi_j) = f_{ij}, \qquad \partial_{y_i}(\varphi_j) = f'_{ij},$$

$$\partial_{x_i}(\psi_j) = g_{ij}, \qquad \partial_{y_i}(\psi_j) = g'_{ij}.$$

Here we may assume that all φ_i and ψ_i have no constant terms. Put

$$Z' = - \sum_{j=1}^{n} (\varphi_j \, \partial_{x_j} + \psi_j \, \partial_{y_j}), \tag{4}$$

then we get

$$[Z', \partial_{v_i}] = D(\partial_{v_i}) \qquad (1 \leq i \leq 2n).$$

<u>Lemma 1</u>. The vector field Z' defined above is hamiltonian modulo linear terms.

<u>Note</u>. The field $Z^{(2)}$ is determined as the component of Z' with coefficient functions of degree ≥ 2. The structure of the linear term of Z', $Z^{(1)} = Z' - Z^{(2)}$, will be studied in §3.3.

<u>Proof</u>. Since $D(\partial_{v_i})$ is hamiltonian for all i, the equalities (1) hold for f_{ij}, g_{ij} and also for f'_{ij}, g'_{ij}. Hence for all i, j, ℓ, we get

$$\partial_{x_\ell}(f'_{ij}) \overset{(2)}{=} \partial_{y_i}(f_{\ell j}) \overset{(1)}{=} \partial_{y_j}(f_{\ell i}) \overset{(2)}{=} \partial_{x_\ell}(f'_{ji}),$$

$$\partial_{y_\ell}(f'_{ij}) = \partial_{y_i}(f'_{\ell j}) = \partial_{y_j}(f'_{\ell i}) = \partial_{y_\ell}(f'_{ji}).$$

and similarly,

$$\partial_{x_\ell}(g_{ij}) \overset{(3)}{=} \partial_{x_i}(g_{\ell j}) \overset{(1)}{=} \partial_{x_j}(g_{\ell i}) \overset{(3)}{=} \partial_{x_\ell}(g_{ji}),$$

$$\partial_{y_\ell}(g_{ij}) = \partial_{x_i}(g'_{\ell j}) = \partial_{x_j}(g'_{\ell i}) = \partial_{y_\ell}(g_{ji}),$$

and further,

$$\partial_{x_\ell}(f_{ij}) \overset{(2)}{=} \partial_{x_i}(f_{\ell j}) \overset{(1)}{=} -\partial_{y_j}(g_{\ell i}) \overset{(3)}{=} -\partial_{x_\ell}(g'_{ji}),$$

$$\partial_{y_\ell}(f_{ij}) = \partial_{x_i}(f'_{\ell j}) = -\partial_{y_j}(g'_{\ell i}) = -\partial_{y_\ell}(g'_{ji}).$$

From the above equalities, we have

$$f'_{ij} \equiv f'_{ji}, \quad g_{ij} \equiv g_{ji}, \quad f_{ij} + g'_{ji} \equiv 0,$$

modulo constant terms for all i, j. This means that Z' satisfies, modulo linear terms, the condition (1) to be hamiltonian.

q. e. d.

3.3. The structure of $Z^{(1)}$.

We have just proved in the preceding paragraph that

$$D(\partial_{v_i}) = [Z', \partial_{v_i}] = [Z^{(1)}, \partial_{v_i}] + [Z^{(2)}, \partial_{v_i}] \qquad (1 \leq i \leq 2n),$$

and that $Z^{(2)}$ is hamiltonian. However $Z^{(1)}$ is not hamiltonian in general. Let us study the linear field $Z^{(1)}$ more in detail.

Put $\quad D' = D - \mathrm{ad}Z^{(2)},\quad$ then we have

$$D'(\partial_{x_i}) = \sum_{k=1}^{n}(a_{ik}\partial_{x_k} + b_{ik}\partial_{y_k}),$$

$$D'(\partial_{y_i}) = \sum_{k=1}^{n}(c_{ik}\partial_{x_k} + d_{ik}\partial_{y_k}),$$

where for all i, k,

$$a_{ik} = f_{ik}(0),\quad b_{ik} = g_{ik}(0),\quad c_{ik} = f'_{ik}(0),\quad d'_{ik} = g'_{ik}(0).$$

Then by (4)

(5) $$Z^{(1)} = -\sum_k\left\{\sum_i(a_{ik}x_i + c_{ik}y_i)\,\partial_{x_k} + \sum_i(b_{ik}x_i + d_{ik}y_i)\,\partial_{y_k}\right\}.$$

Let X_{ij}, Y_{ij} and Z_{ij} be the basis of the linear hamiltonian vector fields, given as

$$X_{ij} = x_i \partial_{x_j} - y_j \partial_{y_i} \qquad (1 \leq i,\ j \leq n),$$

$$Y_{ij} = x_i \partial_{y_j} + x_j \partial_{y_i} \qquad (1 \leq i \leq j \leq n),$$

$$Z_{ij} = y_i \partial_{x_j} + y_j \partial_{x_i} \qquad (1 \leq i \leq j \leq n).$$

Define the functions $\bar{\alpha}_{ijk}$ etc. on U for all i, j, k by

$$D'(X_{ij}) = \sum_k (\bar{\alpha}_{ijk} \partial_{x_k} + \bar{\beta}_{ijk} \partial_{y_k}),$$

$$D'(Y_{ij}) = \sum_k (\bar{\gamma}_{ijk} \partial_{x_k} + \bar{\delta}_{ijk} \partial_{y_k}),$$

$$D'(Z_{ij}) = \sum_k (\bar{\xi}_{ijk} \partial_{x_k} + \bar{\eta}_{ijk} \partial_{y_k}).$$

Then we have the following

Lemma 2. The functions $\bar{\alpha}_{ijk}, \bar{\beta}_{ijk}, \bar{\gamma}_{ijk}, \bar{\delta}_{ijk}, \bar{\xi}_{ijk}$ and $\bar{\eta}_{ijk}$ are of degree ≤ 1, whose linear terms are determined by the constants a_{ik}, b_{ik}, c_{ik} and d_{ik} in (5).

Proof. First, we have for all i, ℓ, m,

$$[\partial_{x_i}, X_{\ell m}] = \delta_{i\ell} \partial_{x_m}, \quad [\partial_{y_i}, X_{\ell m}] = -\delta_{im} \partial_{y_\ell},$$

where δ_{ij} is the Krocker's delta. Applying D' to these equalities, we have

$$\delta_{i\ell} \sum_k (a_{mk}\partial x_k + b_{mk}\partial y_k)$$

$$(6) \qquad = [D'(\partial x_i), X_{\ell m}] + [\partial x_i, D'(X_{\ell m})]$$

$$= a_{i\ell}\partial x_m - b_{im}\partial y_\ell + \sum_k \{\partial x_i(\bar{\alpha}_{\ell mk})\partial x_k + \partial x_i(\bar{\beta}_{\ell mk})\partial y_k\}.$$

$$-\delta_{im} \sum_k (c_{\ell k}\partial x_k + d_{\ell k}\partial y_k)$$

$$(7) \qquad = c_{i\ell}\partial x_m - d_{im}\partial y_\ell + \sum_k \{\partial y_i(\bar{\alpha}_{\ell mk})\partial x_k + \partial y_i(\bar{\beta}_{\ell mk})\partial y_k\}.$$

Compare the coefficients of ∂x_k and ∂y_k, then we see that the derivations of the first order in x_i and y_i of $\bar{\alpha}_{\ell mk}$ and $\bar{\beta}_{\ell mk}$ are constants determined by a_{ij}, b_{ij}, c_{ij} and d_{ij}. Hence we have the assertion for $\bar{\alpha}_{ijk}$ and $\bar{\beta}_{ijk}$.

By the same arguments, we have also the assertion for $\bar{\gamma}_{ijk}$, $\bar{\delta}_{ijk}$, $\bar{\xi}_{ijk}$ and $\bar{\eta}_{ijk}$.

q. e. d.

Lemma 3. There are the following relations:

i). $a_{ij} + d_{ji} = 0 \qquad (i \neq j)$,

ii). $a_{ii} + d_{ii} = a_{jj} + d_{jj}$, iii). $b_{ij} = b_{ji}$ for all i, j.

iv). $c_{ij} = c_{ji}$ for all i, j.

Proof. If $i \neq \ell$ in (6), we have

$$0 = a_{i\ell}\,\partial x_m - b_{im}\,\partial y_\ell + \sum_k\{\partial x_i(\bar{\alpha}_{\ell mk})\,\partial x_k + \partial x_i(\bar{\beta}_{\ell mk})\,\partial y_k\}.$$

and hence

$$\text{(8)}\qquad \partial x_i(\bar{\alpha}_{\ell mm}) = -a_{i\ell}, \qquad \partial x_i(\bar{\beta}_{\ell m\ell}) = b_{im} \qquad (i \neq \ell).$$

Put $i = \ell$ in (6), we have

$$\sum_k(a_{mk}\,\partial x_k + b_{mk}\,\partial y_k)$$
$$= a_{ii}\,\partial x_m - b_{im}\partial y_i + \sum_k\{\partial x_i(\bar{\alpha}_{imk})\,\partial x_k + \partial x_i(\bar{\beta}_{imk})\,\partial y_k\}.$$

which implies that

$$\text{(9)}\qquad \begin{cases} \partial x_i(\bar{\alpha}_{imm}) = a_{mm} - a_{ii}, & \\ \partial x_i(\bar{\alpha}_{imk}) = a_{mk} & (k \neq m), \\ \partial x_i(\bar{\beta}_{imk}) = b_{mk} & (k \neq i). \end{cases}$$

Now if $i \neq m$ in (7), we have

$$0 = c_{i\ell}\,\partial_{x_m} - d_{im}\,\partial_{y_\ell} + \sum_k \{\partial_{y_i}(\bar{\alpha}_{\ell mk})\,\partial_{x_k} + \partial_{y_i}(\bar{\beta}_{\ell mk})\,\partial_{y_k}\},$$

and hence

$$(10) \qquad \partial_{y_i}(\bar{\alpha}_{\ell mm}) = -c_{i\ell}, \qquad \partial_{y_i}(\bar{\beta}_{\ell m\ell}) = d_{im} \qquad (i \neq m).$$

Similarly for $i = m$, we get

$$- \sum_k (c_{\ell k}\,\partial_{x_k} + d_{\ell k}\,\partial_{y_k})$$
$$= c_{i\ell}\,\partial_{x_i} - d_{ii}\,\partial_{y_\ell} + \sum_k \{\partial_{y_i}(\bar{\alpha}_{\ell ik})\,\partial_{x_k} + \partial_{y_i}(\bar{\beta}_{\ell ik})\,\partial_{y_k}\}.$$

which implies that

$$(11) \qquad \begin{cases} \partial_{y_i}(\bar{\beta}_{\ell i\ell}) = d_{ii} - d_{\ell\ell}, \\ \\ \partial_{y_i}(\bar{\alpha}_{\ell ik}) = -c_{\ell k} \qquad (k \neq i). \end{cases}$$

Let us take into consideration the condition (1) that $D'(X_{ij})$'s are hamiltonian, then we have from (9), for $m \neq k$,

$$0 = \partial_{x_i}(\bar{\alpha}_{imk}) + \partial_{y_k}(\bar{\beta}_{imi}) = a_{mk} + d_{km};$$

and from (8) and (11), which means i),

Also we obtain from (9) and (11),

$$0 = \partial_{x_i}(\bar{\alpha}_{ijj}) + \partial_{y_j}(\bar{\beta}_{iji}) = a_{jj} - a_{ii} + d_{jj} - d_{ii},$$

which means ii); and from (8) and (9), for $i \neq \ell$,

$$0 = \partial_{x_i}(\bar{\beta}_{\ell m \ell}) - \partial_{x_\ell}(\bar{\beta}_{\ell mi}) = b_{im} - b_{mi},$$

which means iii); and from (10) and (11), for $i \neq m$,

$$0 = \partial_{y_i}(\bar{\alpha}_{\ell mm}) - \partial_{y_m}(\bar{\alpha}_{\ell mi}) = -c_{i\ell} + c_{\ell i},$$

which means iv). q. e. d.

Remark 1. There is no relation besides i) ∼ iv) among a_{ij}, b_{ij}, c_{ij} and d_{ij}, which comes from the condition that $D'(V)$ is hamiltonian, where V is any one of X_{ij}, Y_{ij} and Z_{ij}. Further more there holds

$$D'(V) \equiv [Z^{(1)}, V],$$

modulo constant terms, where V is as above.

Now we can describe the structure of $Z^{(1)}$.

<u>Lemma 4</u>. The vector field $Z^{(1)}$ is uniquely expressed as

$$Z^{(1)} = Z_1^{(1)} + Z_2,$$

where $Z_1^{(1)}$ is hamiltonian, and for some constant c,

$$Z_2 = c \sum_{i=1}^{n} (x_i \partial_{x_i} + y_i \partial_{y_i}).$$

<u>Proof</u>. Put

$$Z^{(1)} = - \sum_{i=1}^{n} \{f_i(x, y) \partial_{x_i} + g_i(x, y) \partial_{y_i}\},$$

then from (5), we have for $i = 1, \cdots, n$,

$$f_i(x, y) = \sum_{j=1}^{n} (a_{ji} x_j + c_{ji} y_j),$$

$$g_i(x, y) = \sum_{j=1}^{n} (b_{ji} x_j + d_{ji} y_j).$$

Put $c = 2^{-1}(a_{ii} + d_{ii})$, which is independent of i by Lemma 3 ii), and put

$$Z_2 = c\sum_i (x_i \partial_{x_i} + y_i \partial_{y_i}).$$

Then, the remaining term $Z_1^{(1)} = Z^{(1)} - Z_2$ is hamiltonian by the equalities i) $\sim$ iv) in Lemma 3. One can easily see that the decomposition $Z^{(1)} = Z_1^{(1)} + Z_2$ is unique, as far as $Z_1^{(1)}$ is hamiltonian and Z_2 is a scalar multiple of $\sum_i (x_i \partial_{x_i} + y_i \partial_{y_i})$. q. e. d.

3.4. Determination of $Z^{(0)}$.

Let the derivation D'' be $D'' = D - \mathrm{ad}Z'$, then by §3.2. we have

$$D''(\partial_{v_i}) = 0 \qquad (1 \leq i \leq 2n).$$

Let α_{ijk}, β_{ijk}, γ_{ijk}, δ_{ijk}, ξ_{ijk} and η_{ijk} be the constant terms of $\bar{\alpha}_{ijk}$, $\bar{\beta}_{ijk}$, $\bar{\gamma}_{ijk}$, $\bar{\delta}_{ijk}$, $\bar{\xi}_{ijk}$ and $\bar{\eta}_{ijk}$ respectively, then, by Lemma 2, we have for all i, j, k,

$$D''(X_{ij}) = \sum_k (\alpha_{ijk} \partial_{x_k} + \beta_{ijk} \partial_{y_k}),$$

$$D''(Y_{ij}) = \sum_k (\gamma_{ijk} \partial_{x_k} + \delta_{ijk} \partial_{y_k}),$$

$$D''(Z_{ij}) = \sum_k (\xi_{ijk} \partial_{x_k} + \eta_{ijk} \partial_{y_k}).$$

Moreover α_{ijk} etc. are expressed more simply as follows.

Lemma 5.

i). $D''(X_{ij}) = \alpha_i \partial_{x_j} + \beta_j \partial_{y_i}$,

ii). $D''(Y_{ij}) = \alpha_i \partial_{y_j} + \alpha_j \partial_{y_i}$,

iii). $D''(Z_{ij}) = -\beta_i \partial_{x_j} - \beta_j \partial_{x_i}$,

where $\alpha_i = \alpha_{iii}$, $\beta_i = \beta_{iii}$.

Proof. Applying D'' to the both sides of the equality

$$[X_{ij}, X_{\ell m}] = \delta_{j\ell} X_{im} - \delta_{im} X_{\ell j},$$

then we get

$$\delta_{j\ell} \sum_k (\alpha_{imk} \partial_{x_k} + \beta_{imk} \partial_{y_k}) - \delta_{im} \sum_k (\alpha_{\ell jk} \partial_{x_k} + \beta_{\ell jk} \partial_{y_k})$$

$$= \left[\sum_k (\alpha_{ijk} \partial_{x_k} + \beta_{ijk} \partial_{y_k}).\ x_\ell \partial_{x_m} - y_m \partial_{y_\ell}\right]$$

$$+ \left[x_i \partial_{x_j} - y_j \partial_{y_i},\ \sum_k (\alpha_{\ell mk} \partial_{x_k} + \beta_{\ell mk} \partial_{y_k})\right]$$

$$= \alpha_{ij\ell} \partial_{x_m} - \beta_{ijm} \partial_{y_\ell} - \alpha_{\ell mi} \partial_{x_j} + \beta_{\ell mj} \partial_{y_i}.$$

If $j \neq \ell$ and $i \neq m$ in (12), we have

$$0 = \alpha_{ij\ell} \partial_{x_m} - \beta_{ijm} \partial_{y_\ell} - \alpha_{\ell mi} \partial_{x_j} + \beta_{\ell mj} \partial_{y_i},$$

which implies that for $j \neq m$,

$$\alpha_{ij\ell} = 0 \qquad (j \neq \ell),$$

and for $i \neq \ell$,

$$\beta_{ijm} = 0 \qquad (i \neq m).$$

Then if $j = \ell$ and $i \neq m$ in (12), we have

$$\alpha_{imm} \partial_{x_m} + \beta_{imi} \partial_{y_i} = \alpha_{i\ell\ell} \partial_{x_m} + \beta_{\ell m\ell} \partial_{y_i},$$

and hence

$$\begin{cases} \alpha_{imm} = \alpha_{i\ell\ell} = \alpha_{iii}, \\ \beta_{imi} = \beta_{\ell m\ell} = \beta_{mmm}. \end{cases}$$

Put $\alpha_i = \alpha_{iii}$, $\beta_i = \beta_{iii}$, then

$$D(X_{ij}) = \alpha_{ijj}\partial_{x_j} + \beta_{iji}\partial_{y_i} = \alpha_i\partial_{x_j} + \beta_j\partial_{y_i},$$

which is the equality i).

Applying D" also to the both sides of the equality

$$\{X_{ij}, Y_{\ell m}\} = \delta_{j\ell} Y_{im} + \delta_{jm} Y_{i\ell},$$

we get

$$\begin{aligned}
&\delta_{j\ell} \sum_k (\gamma_{imk}\partial_{x_k} + \delta_{imk}\partial_{y_k}) + \delta_{jm}\sum_k(\gamma_{i\ell k}\partial_{x_k} + \delta_{i\ell k}\partial_{y_k}) \\
&= \{\alpha_i\partial_{x_j} + \beta_j\partial_{y_i}, x_\ell\partial_{y_m} + x_m\partial_{y_\ell}\} \\
(13)\quad &\qquad + \{x_i\partial_{x_j} - y_j\partial_{y_i}, \sum_k(\gamma_{\ell mk}\partial_{x_k} + \delta_{\ell mk}\partial_{y_k})\} \\
&= \delta_{j\ell}\alpha_i\partial_{y_m} + \delta_{jm}\alpha_i\partial_{y_\ell} - \gamma_{\ell mi}\partial_{x_j} + \delta_{\ell mj}\partial_{y_i}.
\end{aligned}$$

If $j \neq \ell$ and $j \neq m$ in (13), we obtain

$$0 = -\gamma_{\ell mi}\partial_{x_j} + \delta_{\ell mj}\partial_{y_i},$$

and hence for all ℓ , m, i,

$$\gamma_{\ell mi} = 0,$$

$$\delta_{\ell mi} = 0 \qquad\qquad (j \neq m, \ell).$$

Then put $i \neq j = m = \ell$ in (13), we obtain

$$2 \sum_k \delta_{ijk} \partial_{y_k} = 2\alpha_i \partial_{y_j} + \delta_{jjj} \partial_{y_i},$$

which implies by the symmetry of δ_{ijk} in i and j, that

$$\alpha_i = \delta_{ijj} = \delta_{jij},$$

$$\delta_{jjj} = 2\delta_{iji} = 2\alpha_j.$$

Hence we have

$$D(Y_{ij}) = \delta_{ijj} \partial_{y_j} + \delta_{iji} \partial_{y_i} = \alpha_i \partial_{y_j} + \alpha_j \partial_{y_i},$$

for all i, j, which is the equality ii).

Apply D" to the both sides of the equality

$$[X_{ij}, Z_{\ell m}] = -\delta_{i\ell} Z_{jm} - \delta_{im} Z_{j\ell} .$$

Then we get the equality iii) by the same arguments as for ii).

q. e. d.

Thus we have the hamiltonian vector field $Z^{(0)}$, given as

$$(14) \qquad Z^{(0)} = \sum_{i=1}^{n} (\alpha_i \partial_{x_i} - \beta_i \partial_{y_i})$$

with constants α_i, β_i in Lemma 5, such that for any linear hamiltonian vector field V

$$[Z^{(0)}, V] = D''(V).$$

However this condition determines $Z^{(0)}$ by the following lemma.

Lemma 6. Let V be a constant (hamiltonian) vector field with $[V, X_{ij}] = 0$ for all i, j, then we have that $V = 0$.

Proof. Put

$$V = \sum_{i=1}^{n} (a_i \partial_{x_i} + b_i \partial_{y_i}) \qquad (a_i, b_i \in R),$$

then we have

$$0 = [V, X_{ij}] = a_i \partial_{x_j} - b_j \partial_{y_i}.$$

and hence $a_i = b_i = 0$, for all i. q. e. d.

3.5. Thus we know the vector field Z

$$Z = Z^{(0)} + Z^{(1)} + Z^{(2)},$$

as (4), (5) and (14) such that $D(V) = [Z, V]$ for all V with coefficient functions of degree $\lesssim 1$. Then we must show that D is $\mathrm{ad}Z$ for all hamiltonian vector fields on U. This is established by the following

Lemma 7. Assume that a derivation D vanishes at any X such that its coefficient functions are of degree $\lesssim 1$.

Then D is identically zero on $\mathcal{A}_\omega(U)$.

To prove this, we use the following

Lemma 8. Under the assumption of Lemma 7, $D(X) = 0$, if all coefficient functions of X are of degree 2.

The proof of this lemma will be given in §4.

Proof of Lemma 7. Let $X \in \mathcal{A}_\omega(U)$, then we can show that $D(X)(p) = 0$ for any point $p \in U$. In fact, there is a decomposition of X at p,

$X = X_1 + X_2$ such that the coefficient functions of X_1 are polynomials of degree ≤ 2, and $j^2(X_2)(p) = 0$. Then by Lemma 8 and Proposition 4, we have

$$D(X)(p) = D(X_1)(p) + D(X_2)(p) = 0.$$

q. e. d.

§4. Relations to the formal Lie algebras.

4.1. It is known that the derivation algebra of the following irreducible transitive Lie algebra (T L A) $\mathfrak{g}$ of infinite type:

$$\mathfrak{g} = V^{2n} + \mathfrak{sp}(V) + \mathfrak{sp}(V)^{(1)} + \cdots + \mathfrak{sp}(V)^{(p)} + \cdots ,$$

(for definition, see [5] for example) has the subspace of outer derivations, of dimension 1. In other words, $H^1(\mathfrak{A}_\omega(n) ; \mathfrak{A}_\omega(n)) \cong \mathbb{R}$, where $\mathfrak{A}_\omega(n)$ is the Lie algebra of formal hamiltonian vector fields on $\mathbb{R}^{2n}$ at the origin (for definition, see [3] for example). By some techniques used to prove the above formal theorem, we have another approach to the determination of $Z^{(0)}$, and a proof of Lemma 8.

The constant hamiltonian vector fields form a Euclidean vector space (abelian Lie algebra) V_0 of dimension $2n$, and the linear hamiltonian fields form a vector space $V_1 \cong \mathfrak{sp}(V_0)$, with the natural structure of Lie algebra. Before Lemma 5, we have already proved that $D''(V_1) \subset V_0$. The natural representaion of $\mathfrak{sp}(2n\ ;\ \mathbb{R})$ on $\mathbb{R}^{2n}$ is irreducible, and is given in terms of vector fields as $X(\partial_v) = [\partial_v, X]$, where $X \in V_1 \cong \mathfrak{sp}(2n\ ;\ \mathbb{R})$ and $\partial_v \in V_0 \cong \mathbb{R}^{2n}$.

Thus the linear map (derivation) D'' from V_1 to V_0 is a 1-cocycle of $\mathfrak{sp}(2n\ ;\ \mathbb{R})$ with coefficients in the above representation. Apply to D'' the fundamental vanishing theorem for nontrivial irreducible representations of (finite dimensional) semi-simple Lie algebras (cf. [1]). Then we get a unique vector $v_0 \in V_0$ such that

$$D''(X) = (dv_0)(X) = X(v_0) = [Z^{(0)}, X] \qquad (X \in V_1).$$

Here, $Z^{(0)}$ is the vector field corresponding to the vector v_0, and expressed according to the formula in [1] as follows:

$$Z^{(0)} = (2n+1)^{-1}(2n+2)^{-2}\Big\{\sum_{i,j}[X_{ij}, D''(X_{ji})] + \sum_{i<j}([Y_{ij}, D''(Z_{ij})] + [Z_{ij}, D''(Y_{ij})]) + 4^{-1}\sum_{i}([Y_{ii}, D''(Z_{ii})] + [Z_{ii}, D''(Y_{ii})])\Big\}.$$

(It is not easy to obtain the explicit formula (14) of $Z^{(0)}$ from the above expression of it.)

4.2. Proof of Lemma 8.

The hamiltonian vector fields of homogeneous degree 2, also form a vector space $V_2 \cong \mathfrak{sp}(V_0)^{(1)}$, the first prolongation of $V_1 = \mathfrak{sp}(V_0)$. The natural representation of V_1 on V_2 is given in terms of vector fields as $X(Y) = [Y, X]$ for $X \in V_1$ and $Y \in V_2$. Then it is known by H. Weyl [8] that this representaion is irreducible.

As in the proof of Lemma 2, we have $D(V_2) \subset V_0$. From the assumption of Lemma 8, we see that $D([X, Y]) = [X, D(Y)]$ for $X \in V_1$ and $Y \in V_2$. Then the following diagram is commutative:

$$\begin{tikzcd} V_2 \arrow[r, "D"] \arrow[d, "\mathrm{ad}X"'] \arrow[dr, phantom, "\circlearrowright"] & V_0 \arrow[d, "\mathrm{ad}X"] \\ V_2 \arrow[r, "D"] & V_0 \end{tikzcd} \qquad (X \in V_1).$$

This implies that $\ker D$ is stable under $\mathrm{ad}(V_1)$-actions. Since $\ker D \neq \{0\}$ clearly, it follows from the irreducibility of the representation that $\ker D = V_2$, that is $D = 0$ on V_2.

q. e. d.

Remark 1. This proof is simple and short, but is based upon Weyl's work [8]. We have another proof by elementary calculations. Let us sketch it here.

Take a basis X_{ijk}, Y_{ijk}, Z_{ijk} and W_{ijk} of V_2, as

$$X_{ijk} = x_i x_j \partial_{x_k} - x_j y_k \partial_{y_i} - x_i y_k \partial_{y_j},$$

$$Y_{ijk} = x_k y_i \partial_{x_j} + x_k y_j \partial_{x_i} - y_i y_j \partial_{y_k},$$

$$Z_{ijk} = y_i y_j \partial_{x_k} + y_j y_k \partial_{x_i} + y_k y_i \partial_{x_j},$$

$$W_{ijk} = x_i x_j \partial_{y_k} + x_j x_k \partial_{y_i} + x_k x_i \partial_{y_j}.$$

Define the functions $A_{ijk\ell}$ etc. on U by

$$D(X_{ijk}) = \sum_{\ell} (a_{ijk\ell} \partial_{x_\ell} + a'_{ijk\ell} \partial_{y_\ell}),$$

$$D(Y_{ijk}) = \sum_{\ell} (b_{ijk\ell} \partial_{x_\ell} + b'_{ijk\ell} \partial_{y_\ell}),$$

$$D(Z_{ijk}) = \sum_{\ell} (c_{ijk\ell} \partial_{x_\ell} + c'_{ijk\ell} \partial_{y_\ell}),$$

$$D(W_{ijk}) = \sum_{\ell} (d_{ijk\ell} \partial_{x_\ell} + d'_{ijk\ell} \partial_{y_\ell}).$$

Then all these functions are constants as in the proof of Lemma 2.

Moreover these constants are zero. In fact, firstly we obtain that $a_{ijk\ell} = d_{ijk\ell} = d'_{ijk\ell} = 0$ for all i, j, k, ℓ, by applying D to the both sides of the equality

$$[X_{ijk}, Y_{mn}] = \delta_{mk} W_{ijk} + \delta_{nk} W_{ijn},$$

and by the symmetry of $d'_{ijk\ell}$ in i, j, k.

Secondly we obtain that $b'_{ijk\ell} = c_{ijk\ell} = c'_{ijk\ell} = 0$, by applying D to the equality

$$[Z_{mn}, Y_{ijk}] = \delta_{nk} Z_{ijm} + \delta_{mk} Z_{ijn},$$

and by the symmetry of $c_{ijk\ell}$ in i, j, k.

Finally we get that $a'_{ijk\ell} = b_{ijk\ell} = 0$, by applying D to the equality

$$[Z_{mn}, X_{ijk}] = \delta_{ni} Y_{mkj} + \delta_{nj} Y_{mki} + \delta_{mi} Y_{nkj} + \delta_{mj} Y_{nki},$$

and by the symmetry of $b_{ijk\ell}$ in i, j.

Thus we have that $D = 0$ on V_2.

§5. The cohomology $H^1(\text{Ⓐ}_\omega(M) ; \text{Ⓐ}_\omega(M))$.

5.1. In the preceding two sections, we proved Theorem 5, a local theorem. The following one follows immediately from it.

Theorem 5'. $H^1(\text{Ⓐ}_\omega(U) ; \text{Ⓐ}_\omega(U)) \cong R$.

Now we will give a global theorem on M. Before that, we show a global version corresponding to Theorem 5.

Proposition 6. Let (M, ω) be a symplectic manifold, and D a derivation of hamiltonian vector fields $\text{Ⓐ}_\omega(M)$ on M. Then there exists a vector field Z on M such that

$$D(X) = [Z, X] \qquad \text{for all } X \in \text{Ⓐ}_\omega(M).$$

Proof. Take an atlas $\{U_i, \varphi_i : U_i \longrightarrow R^{2n}\}$ of M such that each U_i is a simply connected domain. Then, by Theorem 5 i) and Remark 1 in §2, we have on each U_i a vector field Z_{U_i} such that $D_{U_i}(X) = [Z_{U_i}, X]$ for any $X \in \text{Ⓐ}_\omega(U_i)$. It follows from $D_{U_i}|_{U_i \cap U_j} = D_{U_j}|_{U_i \cap U_j}$ and the uniqueness that $Z_{U_i}|_{U_i \cap U_j} = Z_{U_j}|_{U_i \cap U_j}$. Hence there is a vector field $Z \in \text{Ⓐ}(M)$ such that $Z|_{U_i} = Z_{U_i}$ for each U_i and that $D(X) = [Z, X]$ for any $X \in \text{Ⓐ}_\omega(M)$.

q. e. d.

Let U be a simply connected domain, and $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ a symplectic coordinate system such that $\omega|_U = \Sigma\, dx_i dy_i$. Then, by Theorem 5 ii), the above vector field Z is represented as $Z = Z_{1U} + Z_{2U}$ on U, where $Z_{1U} \in \text{Ⓐ}_\omega(U)$ and $Z_{2U} = c\Sigma(x_i \partial_{x_i} + y_i \partial_{y_i})$ for some constant c. Then we have the following

Proposition 7. If M is connected, the constant c is independent of the choice of U and $(x_1, \cdots, x_n, y_1, \cdots, y_n)$.

Proof. Since M is connected, it is sufficient to show that the constant c is invariant under any symplectic coordinate transformations of U.

Case 1. (Translations). Let new coordinates $(\bar{x}_i, \bar{y}_i)$ be

$$\bar{x}_i = x_i - a_i, \qquad \bar{y}_i = y_i - b_i \qquad (1 \leq i \leq n),$$

where a_i, b_i are real constants. Then

$$\sum_i (x_i \partial_{x_i} + y_i \partial_{y_i}) = \sum_i (\bar{x}_i \partial_{\bar{x}_i} + \bar{y}_i \partial_{\bar{y}_i}) + \sum_i (a_i \partial_{\bar{x}_i} + b_i \partial_{\bar{y}_i}).$$

Since any constant vector field is hamiltonian, the constant c is left invariant.

<u>Case 2</u>. (Linear transformations). We prove that the constant c is left invariant under any general linear transformation, not necessarily symplectic.

Take an element $g = (g_{ij})$ in $GL(2n ; R)$ and put

$$\bar{v}_i = \sum_{j=1}^{2n} g_{ij} v_j \qquad (1 \leq i \leq 2n).$$

Then

$$\partial_{v_i} = \sum_j \partial_{v_i}(\bar{v}_j) \partial_{\bar{v}_j} = \sum_j g_{ji} \partial_{\bar{v}_j},$$

and hence

$$\sum_i v_i \partial_{v_i} = \sum_i (\sum_\ell (g^{-1})_{i\ell} \bar{v}_\ell)(\sum_j g_{ji} \partial_{\bar{v}_j})$$
$$= \sum_{j,\ell} \delta_{j\ell} \bar{v}_\ell \partial_{\bar{v}_j} = \sum_j \bar{v}_j \partial_{\bar{v}_j}.$$

<u>Case 3</u> (General case). We may assume by Case 1 that a symplectic

coordinate transformation φ has no constant terms. Then the inverse φ^{-1} has also no constant terms. The vector fields Z and Z_1 are transformed into hamiltonian ones by means of φ, and the linear term of the expression $\sum_i (x_i \partial_{x_i} + y_i \partial_{y_i})$ in terms of new coordinates depends only on the linear parts of the transformations φ and φ^{-1}. Hence we see by Case 2 that the constant c is invariant under φ, because the higher terms sum up to hamiltonian vector fields by Theorem 5.

q. e. d.

Corollary. Let Z and c be as above, then $L_Z \omega = 2c\,\omega$ on M.

Proof. We see that for any U,

$$L_Z \omega = L_{Z_{2U}} \omega = 2c\,\omega \qquad \text{on } U.$$

q. e. d.

5.2. Now we can prove our main results.

Theorem 8. Let (M, ω) be a connected symplectic manifold. Then the first cohomology of the Lie algebra $Ⓐ_\omega(M)$ with coefficients in its adjoint representation is of dimension 1 or 0, that is,

$$H^1(Ⓐ_\omega(M) \; ; Ⓐ_\omega(M)) \cong R \quad \text{or} \quad 0.$$

Proof. We can define the homomorphism

$$\phi : Ⓓ_\omega(M) \longrightarrow R,$$

which assigns to a derivation $D \in Ⓓ_\omega(M)$ a constant c by Proposition 7. Let us show that $\ker \phi = Ⓓ^i_\omega(M)$. This means that

$$H^1(Ⓐ_\omega(M) \; ; Ⓐ_\omega(M)) \cong Ⓓ_\omega / Ⓓ^i_\omega \cong R \quad \text{or} \quad 0.$$

Let D and D' be two derivations such that $\phi(D) = \phi(D') = c$, and put $\bar{D} = D - D'$, then $\phi(\bar{D}) = 0$. By Remark 1 in §2 and Theorem 5, $\bar{D}$ is inner on any sufficient small simply connected domain U, that is, there exists a unique hamiltonian vector field Z_U such that $\bar{D}_{|U} = \mathrm{ad}(Z_U)$. From the uniqueness in Theorem 5, and by the same arguments in the proof of Proposition 6, there exists a unique vector field Z whose restriction $Z_{|U}$ is equal to Z_U for each such U. Clearly Z is hamiltonian, and $\vec{D}(X) = [Z, X]$ for all $X \in \mathfrak{A}_\omega(M)$. Hence we have that $\ker \phi \subset \mathfrak{D}^i_\omega(M)$.

On the other hand, the converse inclusion:

$\mathfrak{D}^i_\omega(M) \subset \ker \phi$, is clear. q. e. d.

<u>Theorem 9</u>. Assume that the symplectic form ω of M is exact, or there exists a 1-form θ on M such that $d\theta = \omega$. Then

$$H^1(\mathfrak{A}_\omega(M) \ ; \mathfrak{A}_\omega(M)) \cong R.$$

<u>Proof</u>. Let W be a vector field corresponding to θ with respect to ω, that is, $i_W\omega = \theta$. Then

$$\omega = d\theta = d\, i_W\omega = L_W\omega,$$

and hence W is not hamiltonian. On the other hand,

$$L_{[W,X]}\omega = L_W L_X\omega - L_X L_W\omega = -L_X\omega = 0$$

for all $X \in \textcircled{A}_\omega(M)$, then $[W, X] \in \textcircled{A}_\omega(M)$. Therefore ad W is an outer derivation of $\textcircled{A}_\omega(M)$. q. e. d.

Theorem 10. Assume that the symplectic form ω of M is not exact. Then

$$H^1(\textcircled{A}_\omega(M) ; \textcircled{A}_\omega(M)) = 0.$$

Proof. Let D be a derivation of $\textcircled{A}_\omega(M)$. Then by Proposition 6, there is a unique vector field $Z \in \textcircled{A}(M)$ such that $D = \mathrm{ad}\, Z$, and by Corollary of Proposition 7, $L_Z\omega = c\,\omega$ for some constant c. Assume that $c \neq 0$, then $\omega = c^{-1}d(i_Z\omega)$, or ω is exact. Hence $c = 0$, that is, Z is hamiltonian. Thus all derivations of $\textcircled{A}_\omega(M)$ are inner.

q. e. d.

Summarizing these results, we get the following Main Theorem.

Main Theorem. Let (M^{2n}, ω) be a connected symplectic manifold, then

$$\dim H^1(\textcircled{A}_\omega(M) ; \textcircled{A}_\omega(M)) = 1 \text{ or } 0.$$

Moreover, $H^1(\mathfrak{A}_\omega(M) ; \mathfrak{A}_\omega(M)) \cong R$ if and only if the symplectic form ω is exact.

Remark 1. Let M be a manifold attached with a volume form τ or contact form ⓒ. Then, in stead of $\mathfrak{A}_\omega(M)$, we have a natural subalgebra $\mathfrak{A}_\tau(M)$ or $\mathfrak{A}_{ⓒ}(M)$ consisting of vector fields which preserves τ or ⓒ respectively. It is interesting to obtain the analogous results for these subalgebras. If $n = 1$, the above Main Theorem gives the result for $\mathfrak{A}_\tau(M)$ where M is a 2-dimensional smooth manifold.

Remark 2. The condition of continuity is absent in the definition of cochains of $\mathfrak{A}_\omega(M)$ with coefficients in its adjoint representation, but from the above results all cocycles are continuous.

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Added in proof The author is informed of the following work by the referee;

A.Avez-A.Lichnerowicz-A.Diaz-Miranda: Sur l'algebre des automorphismes infinitesimaux d'une variété symplectique, J. Differential Geometry 9(1974), 1-40.

This paper contains essentially the same result as our main theorem, but the method of its proof is different from ours.

主論文

Cohomologies of Lie algebras of vector fields with coefficients in adjoint representations
Case of Classical Type

By

Yukihiro Kanie

Introduction

Let M be a smooth manifold, and $A(M)$ the Lie algebra of all smooth vector fields on M. Assume that M admits a volume form τ, a symplectic form ω or a contact form θ. Then we have natural Lie subalgebras of $A(M)$ as $A_\tau(M)$, $A'_\tau(M)$, $A_\omega(M)$, $A'_\omega(M)$, $A_\theta(M)$ (see §1.1). These Lie algebras including $A(M)$ itself are called of classical type. Here we are interested in the cohomology $H^*(A;A)$ of the Lie algebra A with coefficients in its adjoint representation.

Calculations of them are not easy in general. But the first cohomology can be calculated rather easily since $H^1(A;A)$ is interpreted in terms of derivations of A. From this point of view F. Takens [5] calculated

$H^1(\mathfrak{A}(M);\mathfrak{A}(M))$ in 1973. Later A.Avez - A.Lichnerowicz - A.Diaz-Miranda [2] and the author [3] calculated $H^1(\mathfrak{A}_\omega(M);\mathfrak{A}_\omega(M))$ of Lie algebra $\mathfrak{A}_\omega(M)$ of hamiltonian vector fields by different methods. In the present paper, we will calculate $H^1(\mathfrak{A};\mathfrak{A})$ for all $\mathfrak{A}$ of classical type. Our results can be summarized as follows.

Main Theorem.

a) Let M be a smooth manifold with a volume element τ, a symplectic structure ω or a contact structure θ, and let $\mathfrak{A}$ be one of $\mathfrak{A}(M)$, $\mathfrak{A}'_\tau(M)$, $\mathfrak{A}'_\omega(M)$ and $\mathfrak{A}_\theta(M)$. Then

$$H^1(\mathfrak{A};\mathfrak{A}) = 0.$$

b) Let M be a connected smooth manifold with a volume element τ or a symplectic structure ω, and $\mathfrak{A}=\mathfrak{A}_\tau(M)$ or $\mathfrak{A}_\omega(M)$ respectively. Then

$$H^1(\mathfrak{A};\mathfrak{A}) \cong \mathbb{R} \quad \text{or} \quad 0.$$

Moreover, $H^1(\mathfrak{A};\mathfrak{A}) \cong \mathbb{R}$ if and only if τ or ω is an exact form on M respectively.

We can reduce the study of derivations of $\textcircled{A}$ to the case where M is flat. Here the notion of localizability of derivations (see §1.2) is essential. A Euclidean space is furnished with the natural structure: the volume form $\tau = dx_1 \cdot \cdots \cdot dx_n$, the symplectic form $\omega = \sum_{i=1}^{n} dx_i dx_{i+n}$ or the contact form $\theta = dx_0 - \sum_{i=1}^{n} x_{i+n} dx_i$. Then we have the main theorem for flat case:

a) Let $\textcircled{A} = \textcircled{A}(\mathbb{R}^n)$, $\textcircled{A}'_\tau(\mathbb{R}^n)$, $\textcircled{A}'_\omega(\mathbb{R}^{2n})$ or $\textcircled{A}_\theta(\mathbb{R}^{2n+1})$. Then

$$H^1(\textcircled{A}\,;\textcircled{A}) = 0$$

b) Let $\textcircled{A} = \textcircled{A}_\tau(\mathbb{R}^n)$ or $\textcircled{A}_\omega(\mathbb{R}^{2n})$. Then

$$H^1(\textcircled{A}\,;\textcircled{A}) \cong \mathbb{R}.$$

The contents of the paper are arranged as follows. In §1, we explain the notion of Lie algebras of vector fields of classical type, and the localizability of derivations of $\textcircled{A}$. We also explain the general scheme to prove the main theorem for flat case.

In §2, the properties of $\textcircled{A}_\theta(M)$ and its derivations are studied. In §3, the main theorem for $\textcircled{A}_\theta(\mathbb{R}^{2n+1})$, the flat case, is proved.

In §4, the properties of $\textcircled{A}_{\tau}(M)$, $\textcircled{A}'_{\tau}(M)$ and their derivations are studied. In §5, the main theorems for $\textcircled{A}_{\tau}(\mathbf{R}^n)$ and $\textcircled{A}'_{\tau}(\mathbf{R}^n)$, the flat case, are proved.

In §6, we reproduce briefly the main theorems for $\textcircled{A}_{\omega}(\mathbf{R}^{2n})$ and $\textcircled{A}'_{\omega}(\mathbf{R}^{2n})$ in this direction.

In §7, we prove Main Theorem for all Lie algebras of vector fields of classical type.

The author expresses his hearty thanks to Professors T.Hirai, T.Morimoto and N.Tatsuuma for their kind advices.

§1. Lie algebras of vector fields of classical type, and their derivation algebras.

1.1. Definition of the Lie algebras. All manifolds, vector fields, forms etc. are assumed to be of C^∞-class. Denote by $\mathfrak{A}(M)$ the Lie algebra of all vector fields on a manifold M.

Let τ be a volume element on M. A vector field X is called volume preserving or conformally volume preserving if $L_X\tau = 0$ or $L_X\tau = c\tau$ for some constant c respectively, where L_X denotes the Lie derivation corresponding to X. We get two natural Lie subalgebras $\mathfrak{A}_\tau(M)$ and $\mathfrak{A}'_\tau(M)$ of $\mathfrak{A}(M)$ defined as

$$\mathfrak{A}_\tau(M) = \{X \in \mathfrak{A}(M) \ ; \ L_X\tau = 0\},$$

$$\mathfrak{A}'_\tau(M) = \{X \in \mathfrak{A}(M) \ ; \ L_X\tau = c\,\tau \text{ for some constant } c\}.$$

Then $\mathfrak{A}_\tau(M) \subset \mathfrak{A}'_\tau(M)$ obviously.

Assume that a manifold M of even dimension is furnished with the symplectic structure ω. Here the symplectic structure ω is by definition a non-degenerate closed 2-form on M. A vector field X is called hamiltonian or conformally hamiltonian if $L_X\omega = 0$ or $L_X\omega = c\,\omega$ for some constant c respectively. Thus we have the following two natural Lie subalgebras of $\textcircled{A}(M)$:

$$\textcircled{A}_\omega(M) = \{X \in \textcircled{A}(M) \ ; \ L_X\omega = 0\},$$

$$\textcircled{A}'_\omega(M) = \{X \in \textcircled{A}(M) \ ; \ L_X\omega = c\,\omega \text{ for some constant } c\}.$$

Then $\textcircled{A}_\omega(M) \subset \textcircled{A}'_\omega(M)$ too.

Assume that a manifold M of odd dimension $2n+1$ is furnished with the contact structure θ, where θ is by definition a 1-form on M such that $\theta \wedge (d\theta)^n$ is a volume form on M. A vector field X is called contact if $L_X\theta = f\theta$ for some function f on M. We denote by $\textcircled{A}_\theta(M)$ the Lie subalgebra consisting of all contact vector fields on M.

Let $\textcircled{A}$ be a Lie algebra of vector fields on a manifold M. We call $\textcircled{A}$ of classical type if it is isomorphic to one of the above six Lie algebras: $\textcircled{A}(M)$, $\textcircled{A}_\tau(M)$, $\textcircled{A}'_\tau(M)$, $\textcircled{A}_\omega(M)$, $\textcircled{A}'_\omega(M)$ or $\textcircled{A}_\theta(M)$. The formal algebras corresponding to them are isomorphic to the classical infinite dimensional Lie algebras of É. Cartan (see Singer-Sternberg [4]).

Let U be an open submanifold of M. Then, replacing M by U, we have naturally the Lie algebra $\mathfrak{A}_U$ according to $\mathfrak{A}$. For instance, $\mathfrak{A}_U = \mathfrak{A}_\tau(U)$ for $\mathfrak{A} = \mathfrak{A}_\tau(M)$. Let r_U be the restriction map on U, then $r_U(\mathfrak{A}) \subset \mathfrak{A}_U$, but they do not coincide with each other in general. We say that $\mathfrak{A}$ has <u>the property (A)</u> if $r_U(\mathfrak{A}_{U'}) = r_U(\mathfrak{A})$ for any two open subsets $U \Subset U'$ of M.

<u>Proposition 1.1</u>. The Lie algebras $\mathfrak{A}(M)$, $\mathfrak{A}_\tau(M)$, $\mathfrak{A}_\omega(M)$ and $\mathfrak{A}_\theta(M)$ have the property (A).

<u>Proof</u>. Let $\mathfrak{A}$ be any one of the above Lie algebras. Then for any open subset U of M, the Lie algebra $\mathfrak{A}_U$ is a module over $C^\infty(U)$.

q. e. d.

<u>1.2. Derivations of $\mathfrak{A}$</u>. Let $\mathfrak{A}$ be a Lie subalgebra of $\mathfrak{A}(M)$. A mapping $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ is called a derivation of $\mathfrak{A}$ if D is $\mathbb{R}$-linear and $D([X, Y]) = [D(X), Y] + [X, D(Y)]$ for all $X, Y \in \mathfrak{A}(M)$. A derivation D is called inner if $D = \mathrm{ad}\, W$ for some W in $\mathfrak{A}$. Denote by $\mathfrak{D}(\mathfrak{A})$ the algebra of all derivations of $\mathfrak{A}$, and by $\mathfrak{D}^i(\mathfrak{A})$ its ideal of all inner derivations of $\mathfrak{A}$. Then we know [3 , §1] that the first cohomology $H^1(\mathfrak{A} ; \mathfrak{A})$ of the Lie algebra $\mathfrak{A}$ with coefficients in its adjoint

representation is realized as

$$H^1(A\,;A) \cong D(A)/D^i(A).$$

A derivation D of A is called <u>local</u> if $D(X)$ vanishes on U for any vector field $X \in A$ zero on an open subset U of M. Moreover a local derivation D is called <u>localizable</u> if for any open subset U of M, there is a derivation D_U of A_U compatible with the restriction map r_U, that is, $D_U \circ r_U = r_U \circ D$. Then we have the following.

Proposition 1.2. If the subalgebra A of $A(M)$ has the property (A), then any local derivation of A is localizable.

Proof. Let D be a local derivation of A and U an open subset of M. For any point p of U and $X \in A_U$, by the property (A), there is $\tilde{X} \in A$ such that $X = \tilde{X}$ on some neighbourhood U' of p. Define the derivation D_U of A_U by $D_U(X)(p) = D(\tilde{X})(p)$, then $D_U(X)(p)$ is well-defined because D is local. q. e. d.

If all derivations of A are localizable, the study of $D(A)$ is reduced in a certain extent to the case where M is flat, that is, M is a Euclidean space $V = \mathbb{R}^n$.

1.3. The flat case. Let $Ⓐ$ be a Lie algebra of classical type of vector fields on a Euclidean space V. The main part of our study of the derivation algebra $Ⓓ(Ⓐ)$ of $Ⓐ \subset Ⓐ(V)$ is to find the vector field $W \in Ⓐ(V)$ such that $D = \mathrm{ad}W$ on $Ⓐ$, and to clarify the property of W. This will be done according to the following three steps:

(I) To find a good finite-dimensional subalgebra $Ⓑ$ of $Ⓐ$ for which the following differential equation

$$\text{(E)} \qquad [W, X] = D(X) \qquad (X \in Ⓑ)$$

has a unique solution $W \in Ⓐ(V)$.

(II) Let $Ⓐ_0$ be the subalgebra of $Ⓐ$ consisting of all elements in $Ⓐ$ whose coefficients are polynomials with respect to the coordinates in V. We wish to show that $[W, X] = D(X)$ for all $X \in Ⓐ_0$.

(III) To show the fact that $D(X)(0) = 0$ if a vector field $X \in Ⓐ$ satisfies $j^r(X)(0) = 0$ for some integer r, independent of X.

Here we apply the following lemma.

Proposition 1.3. Suppose that (I), (II) and (III) are established for a $D \in \mathfrak{D}(\mathfrak{A})$, and that $\mathrm{ad}W(\mathfrak{A}) \subset \mathfrak{A}$ where W is the vector field obtained in (I). Then $D = \mathrm{ad}W$ on $\mathfrak{A}$.

Proof. Put $D' = D - \mathrm{ad}W$, then D' is a derivation of $\mathfrak{A}$, zero on $\mathfrak{A}_0$. A vector field $X \in \mathfrak{A}$ is decomposed for any point $p \in V$ as $X = X_1 + X_2$ such that $X_1 \in \mathfrak{A}_0$ and $j^r(X_2)(p) = 0$, because there exists a coordinate transformation φ with polynomial coefficients such that $\varphi(p) = 0$ and $\varphi^*(\mathfrak{A}) = \mathfrak{A}$. By (II) and (III), we get

$$D'(X)(p) = D'(X_1)(p) + D'(X_2)(p) = 0 + 0 = 0 \qquad (p \in V).$$

Hence $D = \mathrm{ad}W$ on $\mathfrak{A}$. q. e. d.

We also apply the following.

Proposition 1.4. It is sufficient for (III) to prove the following:

(III') If a vector field $X \in \mathfrak{A}$ satisfies $j^r(X)(0) = 0$ for some fixed integer $r \geq 0$, then there exist a finite number of vector fields $Y_1, \ldots, Y_{2q} \in \mathfrak{A}$ such that

$$X = \sum_{i=1}^{q} [Y_i, Y_{i+q}] \quad \text{and} \quad j^1(Y_i)(0) = 0 \qquad (1 \leq i \leq 2q).$$

Proof. We get

$$
\begin{aligned}
D(X)(0) &= \sum_{i=1}^{q} D([Y_i, Y_{i+q}])(0) \\
&= \sum_i [D(Y_i), Y_{i+q}](0) + [Y_i, D(Y_{i+q})](0) \\
&= 0 + 0 = 0.
\end{aligned}
$$

q. e. d.

1.4. In §2, we shall prove that any $D \in \mathfrak{D}(\mathcal{A}_\theta(M))$ is localizable (Corollary 2.5), and show (III'), Proposition 2.6, for $\mathcal{A}_\theta(M)$. In §3, we pass through the steps (I) and (II) in §1.3 above for $\mathcal{A}_\theta(n) = \mathcal{A}_\theta(\mathbb{R}^{2n+1})$, Proposition 3.2 and Lemma 3.4. Moreover we obtain the main theorem for $\mathcal{A}_\theta(n)$, Theorem 3.3.

In §4, we clarify the relations between $\mathcal{A}_\tau(M)$ and $\mathcal{A}'_\tau(M)$, and prove that any $D \in \mathfrak{D}(\mathcal{A}'_\tau(M))$ is local (Proposition 4.4), and any $D \in \mathfrak{D}(\mathcal{A}_\tau(M))$ is localizable (Proposition 4.5). In §4.4, (III') for $\mathcal{A}_\tau(M)$, Proposition 4.6, is proved. In §5, the steps (I) and (II) for $\mathcal{A}'_\tau(n) = \mathcal{A}'_\tau(\mathbb{R}^n)$, Proposition 5.6 and Lemma 5.9, are proved. Moreover we obtain the main theorems for $\mathcal{A}'_\tau(n)$ and $\mathcal{A}_\tau(n)$, Theorems 5.7 and 5.8 respectively.

In §6, we describe the outline of the proof of the main theorems for $\mathcal{A}_\omega(\mathbb{R}^{2n})$, $\mathcal{A}'_\omega(\mathbb{R}^{2n})$ and $\mathcal{A}(\mathbb{R}^n)$ in this direction.

§2. Contact vector fields.

2.1. Properties of contact vector fields. Let (M^{2n+1}, θ) be a contact manifold of dimension $2n+1$. Here we do not need the geometrical meaning of the contact vector fields except the following well-known two lemmata.

Lemma 2.1. Let $\#$ be a mapping from $\textcircled{A}_\theta(M)$ to $C^\infty(M)$, which assigns $X^{\#} = i_X\theta$ to $X \in \textcircled{A}_\theta(M)$, where $i_X\theta$ is the interior product of X and θ. Then the linear mapping $\#$ is bijective.

By this lemma, the inverse $\flat : C^\infty(M) \longrightarrow \textcircled{A}_\theta(M)$ can be defined, and we can introduce the generalized Poisson bracket $((\ ,\ ))$ in $C^\infty(M)$ as follows:

$$((f, g))^\flat = [f^\flat, g^\flat] \qquad \text{for} \quad f, g \in C^\infty(M).$$

In this way, $C^\infty(M)$ becomes a Lie algebra isomorphic to $\textcircled{A}_\theta(M)$ under $\#$.

Lemma 2.2. (Darboux). Around any point p of a contact manifold (M^{2n+1}, θ), there exists a coordinate system $(z, x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that θ is expressed as $\theta = dz - \Sigma_{i=1}^n y_i\, dx_i$.

The mapping $\flat$ and the generalized Poisson bracket are written in this contact coordinate system as

$$(2.1) \qquad f^{\flat} = (f - \sum_{i=1}^{n} y_i f_{y_i})\partial_z - \sum_{i=1}^{n} f_{y_i}\partial_{x_i} + \sum_{i=1}^{n} (f_{x_i} + y_i f_z)\partial_{y_i},$$

and

$$(2.2) \qquad ((f,\ g)) = \{f,\ g\}_{x,y} - f_z(g - \sum_j y_j g_{y_j}) + g_z(f - \sum_j y_j f_{y_j})$$

for any $f,\ g \in C^{\infty}(M)$, where $\{\ ,\ \}_{x,y}$ is the usual Poisson bracket in $x_1, \ldots, x_n, y_1, \ldots, y_n$ variables, that is,

$$\{f,\ g\}_{x,y} = \sum_{i=1}^{n} (f_{x_i} g_{y_i} - f_{y_i} g_{x_i}).$$

Here we have the following.

Proposition 2.3. Let X be a contact vector field on M, and U any open subset of M. Assume that $[X,\ Y] = 0$ on U for any $Y \in \mathfrak{X}_{\theta}(M)$ with support contained in U. Then $X = 0$ on U.

Proof. Suppose $X(p) = 0$ for some point p of U. Let U' be a coordinate neighbourhood of p with contact coordinates $(z, x_1, \ldots, x_n, y_1, \ldots, y_n)$ around p. Since X is contact, for the function $f = X^{\#}$, one of $f(p)$, $f_{x_i}(p)$ or $f_{y_i}(p)$ $(1 \leq i \leq n)$ is not zero by (2.1).

Case 1. The case where $f(p) \neq 0$. Let g be a function whose support is contained in U', and equal to z in a smaller neighbourhood U'' of p. Then we have

$$((f, \bar{g})) = -zf_z + f - \sum_{j=1}^{n} y_j f_{y_j} \qquad \text{in } U''.$$

and so $((f, g))(p) = f(p) \neq 0$. Hence we have by (2.1)

$$[X, g^\flat](p) = ((f, g))^\flat(p) \neq 0.$$

This contracts our assumption that $[X, g^\flat] = 0$.

Case 2. The case where $f_{x_i}(p) \neq 0$ or $f_{y_i}(p) \neq 0$. The same arguments as above are also valid here if we take into account the following equalities:

$$((f, y_i)) = f_{x_i}, \qquad ((x_i, f)) = f_{y_i} + x_i f_z.$$

q. e. d.

Proposition 2.4. Any derivation of $Ⓐ_\theta(M)$ is local..

Proof. Suppose that $X \in Ⓐ_\theta(M)$ is identically zero on an open subset U of M. For any $Y \in \mathcal{A}_\theta(M)$ with support contained in U,

$$[D(X), Y] = D([X, Y]) - [X, D(Y)] = 0 - 0 = 0 \quad \text{on } U.$$

By Proposition 2.3, we get $D(X) = 0$ on U. q. e. d.

Corollary 2.5. Any derivation of $\mathcal{A}_\theta(M)$ is localizable.

Proof. This follows directly from Lemmata 1.1 and 1.2.

q. e. d.

2.2. Proposition 2.6. Let X be a contact vector field on M such that $j^4(X)(p) = 0$ at a point $p \in M$. Then there are a finite number of contact vector fields $Y_1, \ldots, Y_{2q}$ on M, and a neighbourhood U of p in M such that

$$X|_U = \sum_{i=1}^{q} [Y_i, Y_{i+q}]|_U$$

and

$$j^1(Y_i)(p) = 0 \qquad (1 \leq i \leq 2q).$$

Proof. By means of a contact coordinate system $(z, x_1, \ldots, x_n, y_1, \ldots, y_n)$ around p, the vector field X and $f = X^{\#}$ are written as

$$X = h\,\partial_z + \sum_{i=1}^{n} (h^i \partial_{x_i} + h^{i+n} \partial_{y_i}),$$

$$X^{\#} = i_X \theta = h - \sum_{i=1}^{n} y_i h^i.$$

We assume that $j^4(h)(0) = 0$ and $j^3(h^i)(0) = 0$ for all i. Then the assertion follows from the next proposition. q. e. d.

Proposition 2.7. Let f be a function on $\mathbb{R}^{2n+1}$ with $j^4(f)(0) = 0$. Then there are a finite number of functions $g_1, \ldots, g_{2q}$ such that

$$f = \sum_{i=1}^{q} ((g_i, g_{i+q})),$$

and

$$j^1(g_i)(0) = j^1(g_{ix_j})(0) = j^1(g_{iy_j}) = 0 \qquad (1 \leq i \leq 2q,\ 1 \leq j \leq n).$$

Proof. *Case 1.* The case where $f_z = 0$. Assume that $j^3(f)(0) = 0$. Then by Proposition 2 in [3], there are functions $g_1, \ldots, g_{2q}$ such that $g_{iz} = 0$, $j^2(g_i)(0) = 0$ $(1 \leq i \leq 2q)$, and $f = \sum_i \{g_i, g_{q+i}\}_{x,y} = \sum_i ((g_i, g_{q+i}))$.

Case 2. The case where f is written as $f = z^2 h$. Assume that $j^3(f)(0) = 0$, that is, $j^1(h)(0) = 0$. Put

$$g = \int_0^{y_1} h\, dy_1 ,$$

then $j^2(g)(0) = 0$, and

$$((x_1 g,\ z^2)) - ((g,\ x_1 z^2))$$
$$= -z^2 x_1 g_z + 2z(x_1 g - x_1 \sum_j y_j g_{y_j}) - \left\{-z^2 g_{y_1} - x_1 z^2 g_z + 2x_1 z(g - \sum y_j g_{y_j})\right\}$$
$$= z^2 g_{y_1} = z^2 h = f.$$

By the above arguments, we may assume that f is expressed as

$$f = z\, x_1^{p_1} \cdots x_n^{p_n}\, y_1^{q_1} \cdots y_n^{q_n}\, h(x,\ y)$$

with $\sum_{i=1}^{n} (p_i + q_i) \geq 4$.

Case 3. The case where $\sum_i p_i \geq 2$.

a) The case where $p_j \geq 2$ for some j. We may assume that f is written as $f = zx_1^2 h(x,\ y)$. Put $g = \int_0^{y_1} h(x,\ y)dy_1$, then $j^2(g)(0) = 0$, and

$$((x_1 g,\ x_1^2 z)) - ((g,\ x_1^3 z)) = zx_1^2 g_{y_1} = zx_1^2 h = f.$$

b) Assume that $p_1 = p_2 = 1$. Then by means of the following contact transformation φ, this case is reduced to a):

$$\varphi : \begin{cases} \bar{x}_1 = \sqrt{2}^{\,-1}(x_1 + x_2), & \bar{y}_1 = \sqrt{2}^{\,-1}(y_1 + y_2), \\ \bar{x}_2 = \sqrt{2}^{\,-1}(x_1 - x_2), & \bar{y}_2 = \sqrt{2}^{\,-1}(y_1 + y_2), \\ \bar{x}_i = x_i \quad , & \bar{y}_i = y_i \qquad (i \geq 3), \\ \bar{z} = z. & \end{cases}$$

Case 4. The case where $\Sigma\, p_j \leq 1$, that is, $\Sigma\, q_i \geq 3$.

a) The case where $q_j \geq 3$ for some j. We may assume that f is written as $f = z\, y_1^3 h(x, y)$. Put $g = \int_0^{x_1} h(x, y)\, dx_1$, then $j^1(g)(0) = 0$, and

$$3((zy_1 g,\ y_1^3)) - 2((zg,\ y_1^4)) = zy_1^3 g_{x_1} = f.$$

b) The case where $q_j = 2$ for some j. We may assume that f is written as $f = zy_1^2 y_2 h(x, y)$. By means of the above transformation φ, this case is reduced to a), because $3y_1^2 y_2 = \sqrt{2}\, \bar{y}_1^3 - \sqrt{2}\, \bar{y}_2^3 + y_2^3$

c) Assume that $q_1 = q_2 = 1$. Then by means of φ, this case is reduced to b). q. e. d.

We have a corollary of Proposition 2.6.

Corollary 2.8. Let D be a derivation of $\mathcal{Q}_\theta(M)$. If X is a contact vector field on M such that $j^4(X)(p) = 0$ for a point p of M, then $D(X)(p) = 0$.

Proof. This follows directly from Proposition 1.4.

q. e. d.

§3. Derivations of $\mathfrak{A}_\theta(n)$.

3.1. Structure of $\mathfrak{A}_\theta(n)$. We consider the natural contact structure $\theta = dz - \sum_i y_i dx_i$ in a Euclidean space $\mathbb{R}^{2n+1}$. In this section, we will study derivations of the Lie algebra $\mathfrak{A}_\theta(n) = \mathfrak{A}_\theta(\mathbb{R}^{2n+1})$ of contact vector fields on $\mathbb{R}^{2n+1}$. At first, we note the following.

Lemma 3.1. A vector field $X = h^0 \partial_z + \sum_{i=1}^n (h^i \partial_{x_i} + h^{i+n} \partial_{y_i})$ on $\mathbb{R}^{2n+1}$ is contact, if and only if it satisfies the following equalities:

$$(\#)_1 \qquad h^0_{y_i} = \sum_{j=1}^n y_j h^j_{y_i} \qquad (1 \leq i \leq n).$$

$$(\#)_2 \qquad y_i (h^0_z - \sum_{j=1}^n y_j h^j_z) = h^{i+n} - h^0_{x_i} + \sum_{j=1}^n y_j h^j_{x_i} \qquad (1 \leq i \leq n).$$

The coefficient functions h^{i+n} $(1 \leq i \leq n)$ are determined by $h^0, h^1, \dots, h^n$.

Proof. Since X is contact, $L_X \theta = g\theta$ for some function g. The assertion follows easily from this. q. e. d.

Let $\mathfrak{B} = \mathfrak{B}_\theta(n)$ be the Lie subalgebra of $\mathfrak{A} = \mathfrak{A}_\theta(n)$ spanned by

$$\begin{cases} Z = \partial_z, \quad X_i = \partial_{x_i}, \quad Y_i = \partial_{y_i} + x_i \partial_z & (1 \leq i \leq n), \\ I = 2z\partial_z + \sum_{i=1}^n (x_i \partial_{x_i} + y_i \partial_{y_i}). & \end{cases}$$

There hold the following relations among them:

$$[Z,\ X_i] = [Z,\ Y_i] = [X_i,\ X_j] = [Y_i,\ Y_j] = 0,\ [X_i,\ Y_j] = \delta_{ij}Z,$$

$$[Z,\ I] = 2Z,\quad [X_i,\ I] = X_i,\quad [Y_i,\ I] = Y_i \qquad (1 \le i,\ j \le n),$$

where δ_{ij} is Kronecker's delta.

For an integer p, we define the subspace $\mathfrak{A}^p$ of $\mathfrak{A}$ as follows:

$$\mathfrak{A}^p = \{X \in \mathfrak{A}_0\ ;\ [I,\ X] = pX\}$$

where $\mathfrak{A}_0$ is defined in §1.3. We have immediately that $[\mathfrak{A}^p,\ \mathfrak{A}^q] \subset \mathfrak{A}^{p+q}$, and that $\mathfrak{A}_0$ is an algebraic direct sum of $\mathfrak{A}^p$'s. We remark the following facts which will be applied later:

i) $\mathfrak{A}^p = \{0\}$ $\qquad (p \le -3)$,

ii) $\mathfrak{A}^{-2} = \mathbb{R}\cdot Z$,

iii) $\mathfrak{A}^{-1} = \sum_{i=1}^{n} (\mathbb{R}\cdot X_i + \mathbb{R}\cdot Y_i)$.

3.2. Now we will solve the equation (E) for $\mathfrak{A}_\theta(n)$, $\mathfrak{B}_\theta(n)$).

Proposition 3.2. Let D be a derivation of $\mathfrak{A}_\theta(n)$. Then there exists a unique vector field W in $Ⓐ_\theta(n)$ such that

$$\text{(E)} \qquad D(X) = [W, X] \qquad \text{for all } X \in Ⓑ_\theta(n).$$

The proof of this proposition will be given in §3.3. Here we deduce from this proposition the following theorem, a local theorem for contact case.

Theorem 3.3. Let D be a derivation of $Ⓐ_\theta(n)$. Then there exists a unique vector dield W in $Ⓐ_\theta(n)$ such that

$$D(X) = [W, X] \qquad \text{for all } X \in Ⓐ_\theta(n).$$

In other words, any derivation of $Ⓐ_\theta(n)$ is inner.

Proof. To prove this theorem, it is sufficient to show that if D is zero on the subalgebra $Ⓑ_\theta(n)$, then D vanishes on the whole $Ⓐ_\theta(n)$. Its proof is reduced to the next lemma by Proposition 1.3 and Corollary 2.8. q. e. d.

Lemma 3.4. If the derivation D of $Ⓐ = Ⓐ_\theta(n)$ is zero on $Ⓑ = Ⓑ_\theta(n)$, then D is zero on $Ⓐ_0$ for $Ⓐ$.

Proof. Assume that $X \in \textcircled{A}^p$, $p \geq 0$, defined in §3.1. The proof is carried out by induction on p. Let $h^i (0 \leq i \leq 2n)$ be functions on $\mathbb{R}^{2n+1}$ defined as

$$D(X) = h^0 \partial_z + \sum_{i=1}^{n} (h^i \partial_{x_i} + h^{i+n} \partial_{y_i}).$$

Apply D to $[Z, X] \in \textcircled{A}^{p-2}$ and $[X_i, X] \in \textcircled{A}^{p-1} (1 \leq i \leq n)$. Then by the assumption of induction, $[Z, D(X)] = [X_i, D(X)] = 0$, so that $h^i_z = h^i_{x_j} = 0 \qquad (0 \leq i \leq 2n,\ 1 \leq j \leq n)$. Hence, by the equalities $(\#)_2$ in Lemma 3.1, we get that

$$h^{i+n} = 0 \qquad\qquad (1 \leq i \leq n).$$

Apply D to $[Y_i, X] \in \textcircled{A}^{p-1}$, then

$$\begin{aligned} 0 = [Y_i, D(X)] &= \Big[\partial_{y_i} + x_i \partial_z,\ h^0 \partial_z + \sum_{j=1}^{n} h^j \partial_{x_j}\Big] \\ &= (h^0_{y_i} - h^i) \partial_z + \sum_{j=1}^{n} h^j_{y_i} \partial_{x_i}, \end{aligned}$$

so that $h^i = h^0_{y_i}$ and $h^j_{y_i} = 0 \qquad$ for $1 \leq i, j \leq n$.

Hence, by the equalities $(\#)_1$ in Lemma 3.1, we get that

$$h^i = h^0_{y_i} = \sum_{j=1}^{n} y_j h^j_{y_i} = 0 \qquad\qquad (1 \leq i \leq n),$$

and so h^0 is a constant.

Apply D to the both sides of $pX = [I, X]$, then

$$ph^0 \partial_z = [I, D(X)] = [I, h^0 \partial_z] = -2h^0 \partial_z.$$

Since $p \geq 0$ by assumption, we get $h^0 = 0$. Hence $D(X) = 0$.

q. e. d.

3.3. Proof of Proposition 3.2. We consider the equation (E) for $(Ⓐ_\theta(n), Ⓑ_\theta(n))$. Let us construct the vector field W as a sum of $W_1, W_2, W_3, W_4 \in Ⓐ_\theta(n)$ as follows:

a) $D(Z) = [W_1, Z]$;

b) $D(X_i) = [W_1 + W_2, X_i]$, $[W_2, Z] = 0 \qquad (1 \leq i \leq n)$;

c) $D(Y_i) = [W_1 + W_2 + W_3, Y_i]$, $[W_3, Z] = [W_3, X_i] = 0 \qquad (1 \leq i \leq n)$;

d) $D(I) = [W_1 + W_2 + W_3 + W_4, I]$, $[W_4, Ⓐ^p] = 0 \qquad (p \leq -1)$,

where

$$Z = \partial_z,\ X_i = \partial_{x_i},\ Y_i = \partial_{y_i} + x_i \partial_z,\ I = 2z\partial_z + \sum_{i=1}^{n} (x_i \partial_{x_i} + y_i \partial_{y_i}).$$

Then $D = \operatorname{ad} W$ on $Ⓑ_\theta(n)$.

Step I. Construction of W_1. Define the functions h^i on $\mathbb{R}^{2n+1}$ by

$$D(Z) = h^0 \partial_z + \sum_{i=1}^{n} (h^i \partial_{x_i} + h^{i+n} \partial_{y_i}).$$

Put the functions φ_1^i and define the vector field W_1 on $\mathbb{R}^{2n+1}$ as

$$\varphi_1^i = \int_0^z h^i dz \qquad (0 \leq i \leq n),$$

$$\varphi_1^{i+n} = \int_0^z h^{i+n} dz + y_i (h^0(0, x, y) - \sum_{j=1}^{n} y_j h^j(0, x, y)) \quad (1 \leq i \leq n);$$

$$W_1 = -\varphi_1^0 \partial_z - \sum_{i=1}^{n} (\varphi_1^i \partial_{x_i} + \varphi_1^{i+n} \partial_{y_i}).$$

Then W_1 satisfies a). Moreover,

Lemma 3.5. W_1 is a contact vector field, or $W_1 \in \mathcal{A}_\theta(n)$.

Proof. Let us prove for W_1 the equalities $(\#)_1$ and $(\#)_2$ in Lemma 3.1.

$(\#)_1$. Put $\psi_i = \varphi^0_{1y_i} - \sum_{j=1}^{n} y_j \varphi^j_{1y_i}$. Then $\psi_i(0, x, y) = 0$ and

$$\psi_{iz} = \varphi^0_{1y_iz} - \sum_{j=1}^{n} y_j \varphi^j_{1y_iz} = h^0_{y_i} - \sum_{j=1}^{n} y_j h^j_{y_i} = 0$$

by the equalities $(\#)_1$ for $D(Z)$. Hence $\psi_i = 0$ for $1 \leq i \leq n$.

$(\#)_2$. Put $\chi_i = y_i(\sum_{j=1}^{n} y_j \varphi^j_{1z} - \varphi^0_{1z}) - \varphi^0_{1x_i} + \varphi^{i+n}_1 + \sum_{j=1}^{n} y_j \varphi^j_{1x_i}$.

Then $\chi_i(0, x, y) = 0$. Moreover taking into account $\varphi^j_{1z} = h^j$ $(0 \leq j \leq 2n)$, we get

$$\chi_{iz} = y_i(\sum_{j=1}^{n} y_j h^j_z - h^0_z) - h^0_{x_i} + h^{i+n} + \sum_{j=1}^{n} y_j h^j_{x_i} = 0$$

from the equalities $(\#)_2$ for $D(Z)$. Hence $\chi_i = 0$ for $1 \leq i \leq n$.

q. e. d.

Step II. Construction of W_2. Put $D_1 = D - \operatorname{ad} W_1$, then $D_1(Z) = 0$. Define the functions f^j_i on $\mathbb{R}^{2n+1}$ as

$$D_1(X_i) = f^0_i \partial_z + \sum_{j=1}^{n} (f^j_i \partial_{x_j} + f^{j+n}_i \partial_{y_j}) \qquad (1 \leq i \leq n).$$

Apply D_1 to $[Z, X_i] = 0$ and $[X_i, X_k] = 0$, then we have

$$f^0_{iz}\partial_z + \sum_{j=1}^{n} (f^j_{iz} \partial_{x_j} + f^{j+n}_{iz} \partial_{y_j}) = 0,$$

$$(f^0_{kx_i} - f^0_{ix_k}) \partial_z + \sum_{j=1}^{n} \{(f^j_{kx_i} - f^j_{ix_k}) \partial_{x_j} + (f^{j+n}_{kx_i} - f^{j+n}_{ix_k}) \partial_{y_j}\} = 0.$$

Hence

$$f^j_z = 0,\quad f^j_{ix_k} = f^j_{kx_i} \qquad (0 \leq j \leq 2n,\ 1 \leq i,\ k \leq n).$$

Therefore we see easily that there exist functions φ^j_2 $(0 \leq j \leq 2n)$

satisfying

$$\varphi^j_{2z} = 0,\quad \varphi^j_{2x_i} = f^j_i \qquad (1 \leq i \leq n).$$

We put

$$\varphi^j_2(0,\ 0,\ y) = 0 \qquad (0 \leq j \leq n),$$

$$\varphi^{j+n}_2(0,\ 0,\ y) = f^0_j(0,\ 0,\ y) - \sum_{k=1}^n y_k\, f^k_j(0,\ 0,\ y) \qquad (1 \leq j \leq n),$$

and

$$W_2 = -\varphi^0_2 \partial_z - \sum_{j=1}^n (\varphi^j_2 \partial_{x_j} + \varphi^{j+n}_2 \partial_{y_j}).$$

Then the vector field W_2 satisfies b). Furthermore,

Lemma 3.6. W_2 is a contact vector field, or $W_2 \in \mathcal{A}_\theta(n)$.

Proof. Let us check $(\#)_1$ and $(\#)_2$ for W_2.

$(\#)_1$. Put $\psi_i = \varphi^0_{2y_i} - \sum_{j=1}^n y_j \varphi^j_{2y_i}$ $(1 \leq i \leq n)$, then

$\psi_i(0,\ 0,\ y) = 0$; $\psi_{iz} = 0$,

$$\psi_{ix_k} = \varphi^0_{2y_ix_k} - \sum_{j=1}^{n} y_j \varphi^j_{2y_ix_k} = f^0_{y_i} - \sum_{j=1}^{n} y_j f^j_{y_i} = 0$$

by the equalities $(\#)_1$ for $D_1(X_k)$. Hence $\psi_i = 0$ for $1 \leq i \leq n$.

$(\#)_2$. Put $\chi_i = \varphi_2^{i+n} - \varphi^0_{2x_i} + \sum_{j=1}^{n} y_j \varphi^j_{2x_i}$. Then

$$\chi_i(0, 0, y) = 0 \; ; \; \chi_{iz} = 0,$$

$$\chi_{ix_k} = f^{i+n}_{x_k} - f^0_{ix_k} + \sum_{j=1}^{n} y_j f^j_{ix_k} = 0 \qquad (1 \leq i, k \leq n)$$

by the equalities $(\#)_2$ for $D_1(X_k)$ because $f^j_{kz} = 0$. Hence $\chi_i = 0$ for $1 \leq i \leq n$.

q. e. d.

Step III. Construction of W_3. Put $D_2 = D_1 - \text{ad}\, W_2$, then $D_2(Z) = D_2(X_i) = 0$ for $1 \leq i \leq n$. Define the functions g^j_i $(0 \leq j \leq 2n)$ on $\mathbb{R}^{2n+1}$ as

$$D_2(Y_i) = g_i^0 \partial_z + \sum_{j=1}^{n} (g^j_i \partial_{x_i} + g_i^{j+n} \partial_{y_i}) \qquad (1 \leq i \leq n).$$

Apply D_2 to $[Z, Y_i] = 0$ and $[X_k, Y_i] = \delta_{ik} Z$, then

$$g^j_{iz} = g^j_{ix_k} = 0 \qquad (0 \leq j \leq n, \; 1 \leq i, k \leq n);$$

$$g^{j+n} = 0 \qquad (1 \leq j \leq n).$$

Apply D_2 to $[Y_i, Y_k] = 0$, then we have

$$(g^0_{ky_i} + g^k_i - g^i_k - g^0_{iy_k})\,\partial_z + \sum_{j=1}^{n}(g^j_{ky_i} - g^j_{iy_k})\,\partial_{x_j} = 0.$$

Hence,

$$g^0_{ky_i} + g^k_i = g^i_k + g^0_{iy_k},$$

$$g^j_{ky_i} = g^j_{iy_k} \qquad (1 \leq i,\ j,\ k \leq n).$$

By $(\#)_1$ for $D_2(Y_i)$, we get from the second equalities above that

$$g^0_{ky_i} = \sum_{j=1}^{n} y_j g^j_{ky_i} = \sum_{j=1}^{n} y_j g^j_{iy_k} = g^0_{iy_k},$$

and so

$$g^k_i = g^i_k \qquad (1 \leq i,\ k \leq n).$$

By the above equalities, there are unique functions $\varphi^j_3 (1 \leq j \leq 2n)$ such that

$$\varphi^j_{3z} = \varphi^j_{3x_i} = 0, \qquad \varphi^j_{3y_i} = g^j_i \qquad (1 \leq i \leq n),$$

and

$$\varphi_3^j(0) = -g_j^0(0), \ \varphi_3^{j+n}(0) = 0 \qquad (1 \leq i \leq n).$$

Finally there is a unique function φ_3^0 such that

$$\varphi_{3z}^0 = \varphi_{3x_i}^0 = 0, \quad \varphi_{3y_i}^0 = g_i^0 + \varphi_3^i \qquad (1 \leq i \leq n),$$

and $\varphi_3^0(0) = 0$. Put

$$W_3 = -\varphi_3^0 \partial_z - \sum_{j=1}^{n} \varphi_3^j \partial_{x_j},$$

then the vector field W_3 satisfies c). Moreover,

Lemma 3.7. W_3 is a contact vector field, or $W_3 \in \mathcal{D}_\theta(n)$.

Proof. W_3 satisfies trivially the equalities $(\#)_2$ in Lemma 3.1. Let us prove the equalities $(\#)_1$. Put

$$\psi_i = \varphi_{3y_i}^0 - \sum_{j=1}^{n} y_j \varphi_{3y_i}^j \qquad (1 \leq i \leq n),$$

then

$$\psi_i(0) = g_i^0(0) + \varphi_3^i(0) = 0,$$

$$\psi_{iz} = \psi_{ix_k} = 0 \qquad (1 \leq i,\ k \leq n),$$

and by $(\#)_1$ for $D_2(Y_k)$, we get also

$$\psi_{iy_k} = g^0_{iy_k} + \varphi^i_{3y_k} - g^k_i - \sum_{j=1}^{n} y_j g^j_{iy_k}$$
$$= g^i_k - g^k_i = 0.$$

Hence $\psi_i = 0$ for $1 \leq i \leq n$. q. e. d.

Step IV. Construction of W_4. Put $D_3 = D_2 - \mathrm{ad}\, W_3$, then $D_3(A^p) = 0$ for $p \leq -1$. Apply D_3 to the both sides of the equalities

$$[Z, I] = 2Z, \quad [X_i, I] = X_i, \quad [Y_i, I] = Y_i \quad (1 \leq i \leq n),$$

then by the same arguments as in the proof of Lemma 3.4, we get

$$D_3(I) = a\, \partial_z \quad \text{for some constant } a.$$

Put $W_4 = 2^{-1} a\, \partial_z$. Then W_4 is a contact vector field and satisfies d), or

$$[W_4, I] = D_3(I), \quad [W_4, A^p] = 0 \quad (p \leq -1).$$

Lemma 3.0. $W_4 = 2^{-1} a\, \partial_z$ is a unique solution of the equations above.

Proof. As in the proof of Lemma 3.4, we see from the fact

$\left[W_4, Ⓐ^p\right] = 0 \quad (p \leq -1)$ that $W_4 \in Ⓐ_\theta(n)$ must be a constant multiple of ∂_z. Put $W_4 = c\,\partial_z$ for some constant c, then $D_3(I) = \left[W_4, I\right] = 2c\,\partial_z$.

Hence $a = 2c$. q. e. d.

The vector field $W = W_1 + W_2 + W_3 + W_4$ is a required one, and the uniqueness of W is guaranteed by the lemma above. This completes the proof of Proposition 3.2.

§4. Volume preserving vector fields.

4.1. Lie algebras $\mathfrak{A}_\tau(M)$ and $\mathfrak{A}'_\tau(M)$. Let M be a connected manifold of dimension n, and τ a volume element on M. Then we get immediately from the definitions of $\mathfrak{A}_\tau(M)$ and $\mathfrak{A}'_\tau(M)$,

$$[\mathfrak{A}'_\tau(M), \mathfrak{A}'_\tau(M)] \subset \mathfrak{A}_\tau(M),$$

and $\mathfrak{A}_\tau(M)$ is an ideal of codimension ≤ 1 in $\mathfrak{A}'_\tau(M)$. Moreover

Lemma 4.1. $\mathfrak{A}_\tau(M)$ is of codimension 1 in $\mathfrak{A}'_\tau(M)$, if and only if the volume form τ is exact, that is, $\tau = d\sigma$ for some $(n-1)$-form σ on M.

Proof. Let τ be exact, that is, $\tau = d\sigma$ for some $(n-1)$-form σ. Then the equality $i_W \tau = \sigma$ determines a vector field W by the non-degeneracy of τ. Hence,

$$L_W \tau = di_W \tau = d\sigma = \tau,$$

so that W lies in $\mathfrak{A}'_\tau(M)$, but not in $\mathfrak{A}_\tau(M)$.

Let $\mathfrak{A}_\tau(M)$ be of codimension 1 in $\mathfrak{A}'_\tau(M)$. Then there is a vector field X such that $L_X \tau = \tau$. Put $\sigma = i_X \tau$, then $\tau = d\sigma$.

q. e. d.

4.2. Properties of volume preserving vector fields. Let X be a volume preserving vector field on a manifold (M, τ). Then $i_X \tau$ is a closed $(n-1)$-form on M, and so the restriction $r_U(i_X \tau)$ is exact by Poincaré's lemma for a sufficiently small open subsets U of M, that is $r_U(i_X \tau) = d\alpha$ for some $(n-2)$-form α on U. In global, any $(n-2)$-form α on M uniquely determines the vector fields $X = X[\alpha]$ in $\mathfrak{X}_\tau(M)$ by the formula

$$i_X \tau = d\alpha.$$

In a coordinate neighbourhood U with coordinates $(x_1, \ldots, x_n)$ such that $\tau = dx_1 \wedge \cdots \wedge dx_n$ in U, any $(n-2)$-form α is written as

$$\alpha = \sum_{i\,j} f_{ij} \sigma_{ij}$$

where $\sigma_{ij} = dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots dx_n$, and f_{ij} are functions on U for $1 \leq i < j \leq n$. Then we have the following.

Lemma 4.2. For any two functions f and g on U,

$$\left[X[f\sigma_{ij}],\, X[g\sigma_{ij}]\right] = (-1)^{i+j} X\left[\{f,\, g\}_{i,j} \sigma_{ij}\right] \qquad \text{on } U.$$

where $\{\ ,\ \}_{i,j}$ is the Poisson bracket in x_i and x_j, that is,

$$\{f, g\}_{i,j} = f_{x_i} g_{x_j} - f_{x_j} g_{x_i} \qquad (1 \leq i < j \leq n).$$

Proof. We have

$$X[f\sigma_{ij}] = (-1)^{i+j-1} f_{x_j} \partial_{x_i} + (-1)^{i+j} f_{x_i} \partial_{x_j},$$

hence,

$$\begin{aligned}
[X[f\sigma_{ij}], X[g\sigma_{ij}]] &= [f_{x_j} \partial_{x_i} - f_{x_i} \partial_{x_j}, g_{x_j} \partial_{x_i} - g_{x_i} \partial_{x_j}] \\
&= -(\{f, g\}_{ij})_{x_j} \partial_{x_i} + (\{f, g\}_{i,j})_{x_i} \partial_{x_j} \\
&= (-1)^{i+j} X[\{f, g\}_{ij} \sigma_{ij}].
\end{aligned}$$

q. e. d.

4.3. Derivations of $\mathfrak{A}'_\tau(M)$.

Proposition 4.3. Let X be a conformally volume preserving vector field on (M, τ), and U any open subset of M. Assume that $[X, Y] = 0$ on U for all $Y \in \mathfrak{A}_\tau(M)$ with support contained in U, then $X = 0$ on U.

Proof. Let $p \in U$ and U' a coordinate neighbourhood of p in U

with coordinates $(x_1, \dots, x_n)$ around p such that $\tau = dx_1 \wedge \dots \wedge dx_n$ in U'. Denote ∂_{x_i} by ∂_i $(1 \leq i \leq n)$. Put $X = \sum_{i=1}^{n} f_i \partial_i$ for some functions f_i on U'. Since the vector fields $\partial_i \in Ⓐ_\tau(U')$,

$$[\partial_i, X] = \sum_{i=1}^{n} \partial_i(f_j)\partial_j = 0 \qquad (1 \leq i \leq n) \quad \text{in} \quad U',$$

and so $\partial_i(f_j) = 0$ for all i, j.

Since $x_i \partial_j \in Ⓐ_\tau(U')$ $(i \neq j)$,

$$[X, x_i \partial_j] = f_i \partial_j = 0 \qquad \text{in} \quad U',$$

hence all f_i are zero in U'. Therefore $X(p) = 0$ for any $p \in U$.

q. e. d.

Proposition 4.4. Any derivation of $Ⓐ_\tau(M)$ or $Ⓐ'_\tau(M)$ is local.

Proof. By the same arguments as in the proof of Proposition 2.4, the assertion for $Ⓐ'_\tau(M)$ follows from Proposition 4.3. As for $Ⓐ_\tau(M)$, see Lemma 5.5 below.

q. e. d.

Corollary 4.5. Any derivation of $Ⓐ_\tau(M)$ is localizable.

Proof. This follows from Lemmata 1.1 and 1.2. q. e. d.

4.4. Proposition 4.6. Let X be a volume preserving vector field on M such that $j^2(X)(p) = 0$ for some point p of M. Then there are a finite number of volume preserving vector fields $Z_1, \ldots, Z_{2q}$ on M and a neighbourhood U of p such that

$$X_{|U} = \sum_{i=1}^{q} [Z_i, Z_{i+q}]_{|U}$$

and

$$j^1(Z_i)(p) = 0 \qquad (1 \leq i \leq 2q).$$

Proof. Introduce a coordinate system $(x_1, \ldots, x_n)$ around p such that $\tau = dx_1 \wedge \cdots \wedge dx_n$. Then, by the arguments in §4.2, the assertion follows from the next proposition. q. e. d.

Proposition 4.7. Let α be an $(n-2)$-form on $\mathbb{R}^n$ such that $j^3(\alpha)(0) = 0$, then there exist a finite number of $(n-2)$-forms $\beta_1, \ldots, \beta_{2q}$ on $\mathbb{R}^n$ such that

$$X[\alpha] = \sum_{i=1}^{q} [X[\beta_i], X[\beta_{i+q}]]$$

and

$$j^2(\beta_i)(0) = 0 \qquad \text{for } 1 \leq i \leq 2q.$$

Proof. Clearly it is enough to show the assertion for the case

$$\alpha = f(x_1, \ldots, x_n)\, dx_3 \wedge \cdots \wedge dx_n = f\,\sigma_{12}$$

with $j^3(f)(0) = 0$. Such a function f can be written as a finite sum of functions of the following type:

$$f = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} h(x_1, \ldots, x_n)$$

with $\sum_{i=1}^{n} r_i \geq 4$.

Case 1. The case where $r_1 \geq 2$ or $r_2 \geq 2$. We may assume that f is written as $f = x_1^2 h(x_1, \ldots, x_n)$. Put

$$g = 3^{-1} \int_0^{x_2} h(x_1, \ldots, x_n)\, dx_2,$$

then $j^2(g)(0) = 0$, and $\{x_1^3, g\}_{1,2} = 3x_1^2 g_{x_2} = f$, that is, by Lemma 4.2,

$$X[f\,\sigma_{12}] = -\left[X[x_1^3\sigma_{12}],\ X[g\,\sigma_{12}]\right].$$

Case 2. The case where r_1 and $r_2 \leq 1$. Then $\sum_{i=3}^{n} r_i \geq 2$. We may assume that f is written as $f = x_i x_j h(x_1, \ldots, x_n)$ for some $i, j \geq 3$.

Put $g = \displaystyle\int_0^{x_2} h(x_1, \ldots, x_n)\,dx_2$, then $j^2(g)(0) = 0$, and $\{x_1x_ix_j, g\}_{12}$ $= x_ix_jg_{x_2} = f$. Then by Lemma 4.2,

$$X[f\sigma_{12}] = -\left[X[x_1x_ix_j\sigma_{12}], X[g\sigma_{12}]\right]. \qquad \text{q. e. d.}$$

We have a corollary of Proposition 4.6.

Corollary 4.8. Let D be a derivation of $\mathfrak{A}_c(M)$. If X is a volume preserving vector field on M such that $j^2(X)(p) = 0$ for a point p of M, then $D(X)(p) = 0$.

Proof. This follows directly from Proposition 1.4. q. e. d.

§5. Derivations of $\mathfrak{A}_\tau(\mathbb{R}^n)$ and $\mathfrak{A}'_\tau(\mathbb{R}^n)$.

5.1. Structure of $\mathfrak{A}'_\tau(n)$. We consider the natural volume element $\tau = dx_1 \wedge \cdots \wedge dx_n$ in a Euclidean space $\mathbb{R}^n$. In this section, we will study derivations of the Lie algebras $\mathfrak{A}_\tau(n) = \mathfrak{A}_\tau(\mathbb{R}^n)$ and $\mathfrak{A}'_\tau(n) = \mathfrak{A}'_\tau(\mathbb{R}^n)$ of volume preserving and conformally volume preserving vector fields on $\mathbb{R}^n$ respectively. At first, we note the following.

Lemma 5.1. Let $X = \sum_{i=1}^{n} f_i \partial_i$ be a vector field on $\mathbb{R}^n$. Then X is volume preserving if and only if $\sum_{i=1}^{n} \partial_i(f_i) = 0$, and is conformally volume preserving if and only if $\sum_{i=1}^{n} \partial_i(f_i) = c$ for some constant c.

Proof. This follows from direct calculations. q. e. d.

Let $\mathfrak{B} = \mathfrak{B}_\tau(n)$ be the Lie subalgebra of $\mathfrak{A} = \mathfrak{A}'_\tau(n)$ spanned by

$$I = \sum_{i=1}^{n} x_i \partial_i, \qquad X_i = \partial_i \qquad (1 \leq i \leq n).$$

There hold the following relations among them:

$$[X_i, X_j] = 0, \quad [X_i, I] = X_i \qquad (1 \leq i, j \leq n).$$

Here we note that the vector field I is not volume preserving because $L_I \tau = n\tau$, and that

$$\mathfrak{A}'_\tau(n) = \mathfrak{A}_\tau(n) + \mathbb{R}\cdot I.$$

For an integer p, we define the subspace $\mathfrak{A}^p$ of $\mathfrak{A}$ as follows:

$$\mathfrak{A}^p = \left\{X \in \mathfrak{A}_0 \; : \; [I,\ X] = pX\right\},$$

where $\mathfrak{A}_0$ is defined in §1.3. We have immediately that $[\mathfrak{A}^p, \mathfrak{A}^q] \subset \mathfrak{A}^{p+q}$, and that $\mathfrak{A}_0$ is an algebraic direct sum of $\mathfrak{A}^p$'s. Moreover,

i) $\mathfrak{A}^p = \{0\}$ $\qquad (p \leq -2)$,

ii) $\mathfrak{A}^{-1} = \sum_{i=1}^{n} \mathbb{R}\cdot X_i$.

5.2. Relations between $\mathfrak{D}(\mathfrak{A}_\tau(n))$ and $\mathfrak{D}(\mathfrak{A}'_\tau(n))$. First we refer the following results of V.I. Arnold [1].

Lemma 5.2. $\quad [\mathfrak{A}_\tau(n), \mathfrak{A}_\tau(n)] = \mathfrak{A}_\tau(n)$.

Note. This lemma can be also obtained by the analogous arguments as in the proof of Proposition 4.4.

Now, we have the following three lemmata.

Lemma 5.3. $\qquad [\textcircled{A}'_\tau(n), \textcircled{A}'_\tau(n)] = \textcircled{A}_\tau(n)$.

Proof. This follows from the inclusion $[\textcircled{A}'_\tau(n), \textcircled{A}'_\tau(n)] \subset \textcircled{A}_\tau(n)$ and the lemma above. q. e. d.

Lemma 5.4. Let D be a derivation of $\textcircled{A}'_\tau(n)$, then $D(\textcircled{A}_\tau(n)) \subset \textcircled{A}_\tau(n)$.

Proof By Lemma 5.3, a vector field $X \in \textcircled{A}_\tau(n)$ is written as

$$X = \sum_{i=1}^{q} [Y_i, Y_{i+q}]$$

by means of a finite number of $Y_1, \ldots Y_{2q} \in \textcircled{A}'_\tau(n)$. Then we have that

$$D(X) = \sum_{i=1}^{q} ([D(Y_i), Y_{i+q}] + [Y_i, D(Y_{i+q})])$$

is volume preserving, by Lemma 5.3. q. e. d.

5.3. Now we will solve the equation (E) for $(\textcircled{A}'_\tau(n), \textcircled{B}'_\tau(n))$.

Proposition 5.5. Let D be a derivation of $\textcircled{A}'_\tau(n)$. Then there exists a unique vector field W, in $\textcircled{A}'_\tau(n)$ such that

(E) $\qquad D(X) = [W, X] \qquad$ for all $X \in \textcircled{B}'_\tau(n)$.

Proposition 5.6. Let D be a derivation of $\textcircled{A}_\tau(n)$. Then there exists a unique vector W in $\textcircled{A}'_\tau(n)$ such that

(E) $\qquad D(X) = [W, X] \qquad$ for all $X \in \textcircled{B}_\tau(n)$.

where $\textcircled{B}_\tau(n) = \textcircled{A}^{-1} + (\textcircled{A}^{0} \cap \textcircled{A}_\tau(n))$ for $\textcircled{A}^{-1}$ and $\textcircled{A}^{0}$ defined in §5.1.

The proof of these two propositions will be given in §5.4. Here we deduce from these propositions the following theorems, local theorems for the volume preserving case.

Theorem 5.7. Let D be a derivation of $\textcircled{A}'_\tau(n)$. Then there exists a unique vector field W in $\textcircled{A}'_\tau(n)$ such that $D(X) = [W, X]$ for all $X \in \textcircled{A}'_\tau(n)$. In other words, any derivation of $\textcircled{A}'_\tau(n)$ is inner.

Theorem 5.8. Let D be a derivation of $\textcircled{A}_\tau(n)$. Then there exists a unique vector field W in $\textcircled{A}'_\tau(n)$ such that

$$D(X) = [W, X] \qquad \text{for all } X \in \textcircled{A}_\tau(n).$$

In other words, the subalgebra of inner derivations of $\textcircled{A}_\tau(n)$ is of codimension 1 in the derivation algebra of $\textcircled{A}_\tau(n)$.

Proof of Theorem 5.7. It is sufficient to show that if D is zero on the subalgebra $\textcircled{B}'_\tau(n)$, then D vanishes on the whole $\textcircled{A}'_\tau(n)$. Its proof is reduced to the next lemma by Proposition 1.3 and Corollary 4.8.

q. e. d.

Lemma 5.9. If a derivation D of $\textcircled{A} = \textcircled{A}'_\tau(n)$ is zero on

$\mathfrak{B} = \mathfrak{B}'_{\tau}(n)$, then D is zero on $\mathfrak{A}_0$ for $\mathfrak{A}$.

Proof. Assume that $X \in \mathfrak{A}^p$, $p \geq 0$, where $\mathfrak{A}^p$ is defined in §5.1. The proof is carried out by induction on p. Define the functions f_i on $\mathbb{R}^n$ as

$$D(X) = \sum_{i=1}^{n} f_i \, \partial_i .$$

Apply D to $[X_i, X] \in \mathfrak{A}^{p-1}$ $(1 \leq i \leq n)$, then we get

$$[X_i, D(X)] = \sum_{j=1}^{n} \partial_i (f_j) \; \partial_j = 0.$$

Hence all f_i are constants, so that $D(X) \in \mathfrak{A}^{-1}$.

Apply D to the both sides of $pX = [I, X]$, then we get

$$p \, D(X) = [I, D(X)] = - D(X).$$

Since $p \geq 0$ by assumption, $D(X)$ must be zero. q. e. d.

Proof of Theorem 5.8. By Proposition 1.4 and Corollary 4.8, it is sufficient to show that if D is zero on the subalgebra $\mathfrak{B}_{\tau}(n)$, then D vanishes also on $\mathfrak{A}^1$ (defined in §5.1). Here note that $\mathfrak{A}^1$ consists of all volume preserving vector fields whose coefficients are homogeneous polynomials of degree 2.

As in the proof of Lemma 5.9, we get that $D(X) \in \textcircled{A}^{-1}$ for $X \in \textcircled{A}^{1}$. Moreover we see that $[D(X), Y] = D([X, Y])$ for all $Y \in \textcircled{A}^{0} \cap \textcircled{A}_{\tau}(n)$. By simple calculations, we get that $D(X) = 0$ for all $X \in \textcircled{A}^{1}$.

q. e. d.

5.4. Proof of Proposition 5.5. Let us consider the equation (E) for $(\textcircled{A}'_{\tau}(n), \textcircled{B}'_{\tau}(n))$. We construct the vector fields W_1 and $W_2 \in \textcircled{A}'_{\tau}(n)$ as follows:

a) $D(X_i) = [W_1, X_i]$ $\quad (1 \leq i \leq n)$,

b) $D(I) = [W_1 + W_2, I]$, $[W_2, X_i] = 0$ $\quad (1 \leq i \leq n)$,

where $X_i = \partial_i$ $(1 \leq i \leq n)$ and $I = \sum_{i=1}^{n} x_i \partial_i$. Put $W = W_1 + W_2$ then $D = \mathrm{ad}\, W$ on $\textcircled{B}'_{\tau}(n)$.

Step I. Construction of W_1. Define the functions f_{ij} on $\mathbb{R}^n$ as

$$D(X_i) = \sum_{j=1}^{n} f_{ij} \partial_j \qquad (1 \leq i \leq n).$$

Apply D to the both sides of $[X_i, X_k] = 0$, then we have

$$\sum_{j=1}^{n} (\partial_i(f_{kj}) - \partial_k(f_{ij}))\, \partial_j = 0 \qquad (1 \leq i, k \leq n),$$

and so

$$\partial_i(f_{kj}) = \partial_k(f_{ij}) \qquad (1 \leq i, k \leq n).$$

Therefore there exist unique functions φ_j $(1 \leq j \leq n)$ such that

$$\partial_i(\varphi_j) = f_{ij} \qquad\qquad (1 \leq i \leq n)$$

and

$$\varphi_j(0) = 0 \qquad\qquad (1 \leq j \leq n).$$

Put $W_1 = -\sum_{i=1}^{n} \varphi_i \partial_i$, then the vector field W_1 satisfies a).

Moreover,

Lemma 5.10. W_1 is a conformally volume preserving vector field, or $W_1 \in \textcircled{A}'_{L}(n)$.

Proof. Since X_k is volume preserving, then by Lemma 5.4, $D(X_k)$ is volume preserving, that is,

$$\sum_{i=1}^{n} \partial_i(f_{ki}) = 0 \qquad\qquad (1 \leq k \leq n).$$

Put $\psi = \sum_{i=1}^{n} \partial_i(\varphi_i)$, then we have

$$\partial_k(\psi) = \partial_k\left(\sum_{i=1}^{n} f_{ii}\right) = \sum_{i=1}^{n} \partial_i(f_{ki}) = 0 \qquad (1 \leq k \leq n),$$

hence ψ is a constant. Then by Lemma 5.1, W_1 is a conformally volume preserving vector field. q. e. d.

Step II. *Construction of* W_2. Put $D' = D - \mathrm{ad}W_1$, then $D'(\textcircled{A}^{-1}) = 0$.

Define the functions g_i on $\mathbf{R}^n$ as

$$D'(I) = \sum_{i=1}^{n} g_i \partial_i .$$

Apply D' to the both sides of $[X_j, I] = X_j$, then we see as in the proof of Lemma 5.9 that all g_i are constants.

Put $W_2 = \sum_{i=1}^{n} g_i \partial_i = \sum_{i=1}^{n} g_i(0) \partial_i$. Then W_2 is a volume preserving vector field and satisfies b), or

$$[W_2, I] = D'(I), \qquad [W_2, X_i] = 0 \qquad (1 \leq i \leq n).$$

Lemma 5.11. $W_2 = \sum_i g_i \partial_i$ is a unique solution of the equations above.

Proof. As in the proof of Lemma 5.9, we see from $[W_2, \textcircled{A}^{-1}] = 0$ that W_2 must be a vector field with constant coefficients. Put $W_2 = \sum_{i=1}^{n} a_i \partial_i$, then

$$D'(I) = [W_2, I] = \sum_{i=1}^{n} a_i \partial_i .$$

Hence $a_i = g_i$ for $1 \leq i \leq n$. q. e. d.

The vector field $W = W_1 + W_2$ is a required one, and the uniqueness of W is guaranteed by the lemma above. This completes the proof of Proposition 5.6.

Proof of Proposition 5.6. It is sufficient to construct uniquely the vector fields $W_1, W_2 \in \textcircled{A}'_{\tau}(n)$ as follows:

a) $D(X) = [W_1, X]$ $\qquad (X \in \textcircled{A}^{-1})$

b) $D(Y) = [W_1 + W_2, Y], \ [W_2, X] = 0$ $\qquad (Y \in \textcircled{A}^{0} \cap \textcircled{A}_{\tau}(n))$.

The construction of W_1 is exactly the same as in the proof of Proposition 5.5. And one can construct easily a unique W_2 by the similar way as in the hamiltonian case [3]. q. e. d.

§6. Remarks on derivations of $\textcircled{A}'_{\omega}(M)$ and $\textcircled{A}'_{\omega}(n)$.

6.1. Hamiltonian vector fields. Let (M,ω) be a connected symplectic manifold. By the analogous arguments as in §4, we get the following propositions.

Lemma 6.1. $\textcircled{A}_{\omega}(M)$ is an ideal of codimension ≤ 1 in $\textcircled{A}'_{\omega}(M)$. Moreover the codimension equals to one, if and only if the symplectic form ω is an exact 2-form.

Proposition 6.2. Any derivation of $\textcircled{A}'_{\omega}(M)$ is local.

Proposition 6.3. Any derivation of $\textcircled{A}_{\omega}(M)$ is localizable.

Since Proposition 1 in [3] is nothing but the assertion (III') for $\textcircled{A}_{\omega}(M)$, we get by Proposition 1.4 the following

Proposition 6.4. Let D be a derivation of $\textcircled{A}_{\omega}(M)$. If X is a hamiltonian vector field on M such that $j^2(X)(p) = 0$ at a point $p \in M$, then $D(X)(p) = 0$.

6.2. Derivations of $\textcircled{A}_{\omega}(n)$ and $\textcircled{A}'_{\omega}(n)$. By the similar method as for the volume preserving case, one can reproduce Theorem 5 in [3].

a local theorem for the hamiltonian case. Let us sketch it here for completeness.

We consider the natural symplectic structure $\omega = \sum_i dx_i dx_{i+n}$ on a Euclidean space $\mathbb{R}^{2n}$, then we get the following three lemmata similarly as in §5.2.

Lemma 6.5 (cf. [1]).

$$[Ⓐ'_\omega(n), Ⓐ'_\omega(n)] = [Ⓐ_\omega(n), Ⓐ_\omega(n)] = Ⓐ_\omega(n) = Ⓐ_\omega(\mathbb{R}^{2n}).$$

Lemma 6.6. Let D be a derivation of $Ⓐ'_\omega(n) = Ⓐ'_\omega(\mathbb{R}^{2n})$. Then $D(Ⓐ_\omega(n)) \subset Ⓐ_\omega(n)$.

Let $Ⓑ = Ⓑ_\omega(n)$ be the Lie subalgebra of $Ⓐ = Ⓐ'_\omega(n)$ spanned by

$$I = \sum_{i=1}^{2n} x_i \partial_i, \quad X_i = \partial_i \qquad (1 \leq i \leq 2n).$$

Note that $L_I \omega = 2\omega$, then we get

$$Ⓐ'_\omega(n) = Ⓐ_\omega(n) + \mathbb{R} \cdot I.$$

For an integer p, the subspace A^p of A is defined as

$$A^p = \{X \in A_0 \ ; \ [I, X] = pX\}.$$

We can solve the equation (E) for $(A'_\omega(n), B_\omega(n))$.

Proposition 6.7. Let D be a derivation of $A'_\omega(n)$. Then there exists a unique conformally hamiltonian vector field W on $\mathbb{R}^{2n}$ such that

$$\text{(E)} \qquad D(X) = [W, X] \qquad \text{for all } X \in B_\omega(n).$$

Outline of Proof. The proof is almost the same as the proof of Proposition 5.5. The vector field W is determined by the values of D at X_i $(1 \leq i \leq 2n)$ up to constant vector fields (Step I). The value $D(I)$ determines the constant terms of W (Step II). We see similarly as Lemma 5.10 that $W' = W - W_1$ is hamiltonian, where W_1 is the linear term of W, A^0-component of W. Applying the derivation $D - \mathrm{ad}W'$ to $[X_i, A^0] \subset A^{-1}$ $(1 \leq i \leq 2n)$, we see that $D - \mathrm{ad}W' = \mathrm{ad}W_1$ on A^p $(p \leq 0)$ and that W_1 is conformally hamiltonian.

We get from Propositions 6.4 and 6.7 the following theorem analogously as Theorems 5.7 and 5.8.

Theorem 6.8 (Theorem 5 in [3]). Let D be a derivation of $\mathfrak{A}_\omega(n)$ or $\mathfrak{A}'_\omega(n)$. Then there exists a unique conformally hamiltonian vector field $W \in \mathfrak{A}'_\omega(n)$ on R^{2n} such that $D = \mathrm{ad}W$.

6.3. The results on the derivation of $\mathfrak{A}(n) = \mathfrak{A}(R^n)$ in the paper [5] of F. Takens can be obtained more simply in this direction. Let $\mathfrak{B} = \mathfrak{B}(n)$ be the Lie subalgebra of $\mathfrak{A} = \mathfrak{A}(n)$ spanned by

$$I = \sum_{i=1}^{n} x_i \partial_i, \qquad X_i = \partial_i \qquad (1 \leq i \leq n).$$

For an integer p, define the subspace $\mathfrak{A}^p$ of $\mathfrak{A}(n)$ as

$$\mathfrak{A}^p = \{X \in \mathfrak{A}_0 ; [I, X] = pX\}.$$

Then we get

Theorem 6.9 (Lemma 4 in [5]). Let D be a derivation of $\mathfrak{A}(n)$. Then there exists a unique vector field W on R^n such that $D = \mathrm{ad}W$ on $\mathfrak{A}(n)$.

Key of Proof. The vector field W is determined by the values $D(X_i)$ $(1 \leq i \leq n)$ and $D(I)$.

§7. The cohomology $H^1(\mathfrak{A};\mathfrak{A})$.

7.1. The main theorem for flat cases. The following main theorem for flat cases is obtained immediately from Theorems 3.3, 5.7, 5.8, 6.8 and 6.9 for respective Lie algebras of classical type.

Theorem 7.1. a) Let $\mathfrak{A}=\mathfrak{A}(\mathbb{R}^n), \mathfrak{A}'_\tau(\mathbb{R}^n), \mathfrak{A}'_\omega(\mathbb{R}^{2n})$ or $\mathfrak{A}_\theta(\mathbb{R}^{2n+1})$. Then

$$H^1(\mathfrak{A};\mathfrak{A}) = 0.$$

b) Let $\mathfrak{A}=\mathfrak{A}_\tau(\mathbb{R}^n)$ or $\mathfrak{A}_\omega(\mathbb{R}^{2n})$. Then

$$H^1(\mathfrak{A};\mathfrak{A}) \cong \mathbb{R}.$$

Here

$$\tau = dx_1 \cdots dx_n, \quad \omega = \sum_{i=1}^{n} dx_i dx_{i+n}, \quad \theta = dx_0 - \sum_{i=1}^{n} x_{i+n} dx_i.$$

7.2. Main Theorem. Let M be a smooth manifold with a volume element τ, a symplectic structure ω or a contact structure θ, and let $\mathfrak{A}$ be one of $\mathfrak{A}(M), \mathfrak{A}'_\tau(M), \mathfrak{A}'_\omega(M)$ and $\mathfrak{A}_\theta(M)$. Then

$$H^1(\mathcal{A};\mathcal{A}) = 0.$$

b) Let M be a connected smooth manifold with a volume element τ or a symplectic structure ω, and $\mathcal{A} = \mathcal{A}_\tau(M)$ or $\mathcal{A}_\omega(M)$ respectively. Then

$$H^1(\mathcal{A};\mathcal{A}) \cong \underset{\sim}{R} \quad \text{or} \quad 0.$$

Moreover, $H^1(\mathcal{A};\mathcal{A}) \cong \underset{\sim}{R}$ if and only if τ or ω is an exact form on M respectively.

7.3. Proofs for $\mathcal{A}(M)$ and $\mathcal{A}_\theta(M)$. Let us prove that any derivation D of $\mathcal{A}$ is inner. Take an atlas $\{U_i, \varphi_i\}_{i\in I}$ such that each U_i are connected and simply connected. Since D is localizable, the derivation D_{U_i} of $\mathcal{A}_{U_i}$ can be defined for all $i \in I$ in such a way that $r_{U_i} \circ D = D_{U_i} \circ r_{U_i}$. Then by Theorems 3.3 and 6.9 in respective cases, there exists for any $i \in I$ a unique vector field $W_i \in \mathcal{A}_{U_i}$ such that $D_{U_i} = \mathrm{ad}W_i$ on $\mathcal{A}_{U_i}$. Since $D_{U_i} \circ r_{U_i \cap U_j} = D_{U_j} \circ r_{U_i \cap U_j}$, we get $r_{U_i \cap U_j}(W_i) = r_{U_i \cap U_j}(W_j)$ by the uniqueness of W_U. Hence there exists a vector field $W \in \textcircled{A}$ such that $r_{U_i}(W) = W_i$ for all $i \in I$ and that $D = \mathrm{ad}\, W$ on $\textcircled{A}$. q. e. d.

7.4. Proof for $\textcircled{A}_\tau(M)$ and $\textcircled{A}_\omega(M)$. Here we denote $\textcircled{A}_\tau(M)$ or $\textcircled{A}_\omega(M)$ by $\textcircled{A}$, and $\textcircled{A}'_\tau(M)$ or $\textcircled{A}'_\omega(M)$ by $\textcircled{A}'$ respectively.

Lemma 7.2. For any $X \in \textcircled{A}'$, $\mathrm{ad}\, X$ is a derivation of $\textcircled{A}$.

Proof. Let σ be τ or ω, then

$$L_{[X,Y]}\sigma = L_X L_Y \sigma - L_Y L_X \sigma = 0 \qquad (Y \in \textcircled{A}).$$

q. e. d.

Let D be a derivation of $\textcircled{A}$. Since D is localizable, for any open subset U of M, the derivation D_U of $\textcircled{A}_U$ can be defined in such a way that $r_U \circ D = D_U \circ r_U$. Then by Theorems 5.8 and 6.8 in respective cases, we get a unique vector field W_U of $\textcircled{A}'_U$ such that $D_U = \mathrm{ad}\, W_U$ on $\textcircled{A}_U$ for any sufficiently small U. By the arguments in §7.3, there is a vector field $W \in \textcircled{A}'$ such that $r_U(W) = W_U$ and that $D = \mathrm{ad}\, W$ on $\textcircled{A}$. Hence by Lemma 7.2, we get the isomorphism $D(\textcircled{A}) \cong \textcircled{A}'$. Therefore the assertion follows from Lemmata 4.1 and 6.1 in respective cases.

q. e. d.

7.5. Proofs for $\textcircled{A}'_\tau(M)$ and $\textcircled{A}'_\omega(M)$. Here we use the notations $\textcircled{A}$

and $\mathfrak{A}'$ as in §7.4. Let D' be a derivation of $\mathfrak{A}'$, then $D = D'|_{\mathfrak{A}}$ is a derivation of $\mathfrak{A}$ with values in $\mathfrak{A}'$. Since D is localizable, for any open subset U of M, the derivation D_U of $\mathfrak{A}_U$ with values in $\mathfrak{A}'_U$ can be defined in such a way that $r_U \circ D = D_U \circ r_U$, as in the proof of Proposition 1.2. If U is sufficiently small, $D_U(\mathfrak{A}_U) \subset \mathfrak{A}_U$ by the same arguments as in the proof of Lemma 5.4. Then by Theorems 5.8 and 6.8 in respective cases, we get a unique vector field $W_U \in \mathfrak{A}'_U$ such that $D_U = \operatorname{ad} W_U$ on $\mathfrak{A}_U$. By the arguments in §7.3, there is a vector field $W \in \mathfrak{A}'$ such that $r_U(W) = W_U$ and that $D = \operatorname{ad} W$ on $\mathfrak{A}$.

For any $Y \in \mathfrak{A}'$ and all $X \in \mathfrak{A}$, we get

$$\begin{aligned}
[D'(Y), X] &= D([Y, X]) - [Y, D(X)] \\
&= [W, [Y, X]] - [Y, [W, X]] \\
&= [[W, Y], X].
\end{aligned}$$

By Proposition 4.3 and the similar proposition for the hamiltonian case, we see

$$D'(Y) = [W, Y] \qquad (Y \in \mathfrak{A}').$$

Thus any derivation D' is inner. q. e. d.

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